Distributivity in $Ł_{\aleph_0}$ and Other Sentential Logics

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Abstract. Certain distributivity results for Łukasiewicz's infinite-valued logic $Ł_{\aleph_0}$ are proved axiomatically (for the first time) with the help of the automated reasoning program OTTER [16]. In addition, *non*-distributivity results are established for a wide variety of positive substructural logics by the use of logical matrices discovered with the automated model finding programs MACE [15] and MAGIC [25].

Keywords: distributivity, sentential, logic, Łukasiewicz, detachment, substructural

1. Motivation

This research was originally motivated by the following uncharacteristic remark made by Rose and Rosser in their classic paper on Łukasiewicz's infinite-valued logic $Ł_{\aleph_0}$ [23, pp. 11-12]:

> With both A and B serving as disjunctions and both K and L serving as conjunctions, one can write a number of possible distributive laws. Some are not valid, and of the valid ones we have been able to prove only two from the axiom schemes A1-A4.

We took up the challenge to prove the valid $Ł_{\aleph_0}$ distributivity results that Rose and Rosser had been unable to establish axiomatically.[1] Our success lead us to a much deeper understanding of distributivity, not only in $Ł_{\aleph_0}$, but in a wide variety of positive sentential logics, both classical and non-classical. Much of what we learned is reported here.

2. Polish Notation and Other Conventions

We will adopt the well-known Polish notation of Łukasiewicz. In particular, 'C' stands for implication, 'N' for negation, 'A' for disjunction, and 'K' for conjunction. Lower case letters 'p', 'q', etc., are used as sentential or propositional variables. To illustrate how Polish notation differs from infix and prefix notation (both of which are supported by OTTER), we show how a distributive law looks in each of the three:

Polish:	$CKpAqrAKpqKpr$
Infix:	$(p \,\&\, (q \vee r)) \rightarrow ((p \,\&\, q) \vee (p \,\&\, r))$
Prefix:	C(K(p, A(q, r)), A(K(p, q), K(p, r)))

[1] Note added in proof: Beavers [2] has independently taken up this challenge.

We choose Polish notation not just because of its historical dominance in this area, but also because of its syntactic economy. Some of the proofs we will present are quite complex and would be difficult to parse and/or typeset in either the infix or the prefix notation used by OTTER.

The proofs presented in this paper have been translated directly from OTTER output files into our Polish notation.[2] Hyperresolution is used, throughout, to implement all logical rules of inference in the systems presented. The notation '[i, j, k]' written to the left of each formula in our proofs is shorthand for 'the present formula was obtained by applying the rule at line i (*via* a single hyperresolution step), with the formula at line j as major premise and the formula at line k as minor premise.' Rules of inference are stated using commas to separate the premises of the rule, and a double arrow '$\Rightarrow$' to indicate its conclusion. For instance, *modus ponens* (or detachment) looks like: '$Cpq, p \Rightarrow q$'.

3. Łukasiewicz's Infinite-Valued Logic $Ł_{\aleph_0}$

3.1. The Semantics of $Ł_{\aleph_0}$

The semantics of the infinite-valued logic $Ł_{\aleph_0}$ were first presented by Łukasiewicz [13, pp. 129-130]. His original motivation was to provide a "numerical interpretation" of the sentential variables which appear in Whitehead and Russell's *Principia Mathematica*. This numerical interpretation was intended to (i) provide a way to show that "some logical laws are independent of the others," and (ii) to extend Łukasiewicz's three-valued logic [13, pp. 87-109] to a logic with infinitely many values or "degrees of probability corresponding to various possibilities."[3]

We find Łukasiewicz's original presentation of the semantics somewhat cumbersome to apply. What follows is Wajsberg's [28, p. 123, note 8] simplification of the semantics for $Ł_{\aleph_0}$. We introduce a *valuation function* v which assigns rational numbers on the closed unit interval $[0, 1]$ to each atomic sentence in the language of $Ł_{\aleph_0}$. The values v assigns to complex $Ł_{\aleph_0}$ sentences are determined recursively, as follows:

$$v(Cpq) = q \dot{-} p = \begin{cases} 0 & \text{if } p \geq q, \\ q - p & \text{if } p < q. \end{cases} \qquad v(Kpq) = \max(p, q)$$

$$v(Apq) = \min(p, q) \qquad v(Np) = 1 - p$$

[2] See www.mcs.anl.gov/~fitelson/distrib/ for relevant input files.

[3] See [11, pp. 569–572] for the historical development of the thoughts of Łukasiewicz and his contemporaries on many-valued logics. See [26] for a recent survey of interpretations and applications of many-valued logics, including $Ł_{\aleph_0}$.

In other words, v assigns a conditional of the form Cpq the value $q \dot{-} p$, where $\dot{-}$ is *cutoff subtraction*, v assigns a conjunction Kpq the *maximum* of the values of p and q, v assigns a disjunction Apq the *minimum* of the values of p and q, and v assigns a negation Np the value $1 - p$.[4] Thinking in terms of cutoff subtraction, max, and min, makes calculating the values of complex $Ł_{\aleph_0}$ sentences relatively easy.

There is just one designated value in this semantics: *zero*. For any sentence p of $Ł_{\aleph_0}$, p is said to be *semantically valid in* $Ł_{\aleph_0}$ just in case $v(p) = 0$, for all valuations v. As an illustration, we now present a proof that the following distributive law is semantically valid in $Ł_{\aleph_0}$.

(1) $$CKpAqrAKpqKpr$$

The proof that (1) is valid in $Ł_{\aleph_0}$ involves calculating $v[(1)]$ in six cases, corresponding to all of the different ways in which p, q, and r can be partially ordered, according to any possible valuation v.[5] To this end, we present the following "$Ł_{\aleph_0}$ truth-table" for the formula (1).

Six Cases	$v(Aqr)$	$v(KpAqr)$	$v(Kpq)$	$v(Kpr)$	$v(AKpqKpr)$	$v[(1)]$
$p \le q \le r$	q	q	q	r	q	0
$p \le r \le q$	r	r	q	r	r	0
$q \le p \le r$	q	p	p	r	p	0
$q \le r \le p$	q	p	p	p	p	0
$r \le p \le q$	r	p	q	p	p	0
$r \le q \le p$	r	p	p	p	p	0

Proving that the Distributive Law (1) is Valid in $Ł_{\aleph_0}$

Inspection of the above table reveals that every possible valuation v is such that $v[(1)] = 0$. Therefore, (1) is semantically valid in $Ł_{\aleph_0}$. In the sections that follow, we will provide various axiomatic proofs of (1). An axiomatic proof of (1) eluded Rose and Rosser [23].

[4] Strictly speaking, we should always write '$v(p)$' when we talk about the value v assigns to the sentence p. But, we will, for simplicity, often write things like '$p \le q$', where it is understood in such cases that 'p' and 'q' denote the *numerical values* that v assigns to the sentences p and q, respectively (not the sentences themselves).

[5] Wajsberg [28, p. 125, Theorem 17] proves that the set of valid $Ł_{\aleph_0}$ sentences is decidable. The validity of (1) is equivalent to a conjunction of six statements in the first-order theory of the rationals. For example, the first case can be rephrased as:

$$(\forall p)(\forall q)(\forall r)[p \le q \,\&\, q \le r \Rightarrow \max(p, \min(q, r)) \dot{-} \min(\max(p, q), \max(p, r)) = 0]$$

Generally, the $Ł_{\aleph_0}$-validity of a sentence p is equivalent to a conjunction of universally quantified Boolean combinations of linear equalities and inequalities. For this class of first-order formulas, Wajsberg proves a quantifier elimination theorem, along the lines of Langford's [12] decidability proof for the theory of dense linear orders.

3.2. The Axiomatics of $Ł_{\aleph_0}$

Łukasiewicz [13, pp. 143-144] conjectured that the following set of five[6] axiom schemes (A1–A5) and the single rule (CD) of detachment[7] is *complete* with respect to the semantics of $Ł_{\aleph_0}$ described above.

(CD) $Cpq, p \Rightarrow q$

(A1) $CpCqp$

(A2) $CCpqCCqrCpr$

(A3) $CCCpqqCCqpp$

(A4) $CCNpNqCqp$

(A5) $CCCpqCqpCqp$

Łukasiewicz's conjecture is rumored to have been proven first by his student Mordchaj Wajsberg [13, p. 144] [28, p. 105]. However, Wajsberg's proof was never made public. Rose and Rosser [23] were the first to publish a proof of the completeness of A1–A5. Since then, others have provided completeness proofs with respect to more general classes of algebraic structures [6, 7].

In $Ł_{\aleph_0}$, A and K are defined in terms of C and N, as follows:

$$Apq =_{df} CCpqq$$
$$Kpq =_{df} NCCNpNqNq$$

In the next section, we will report an OTTER proof of the distributivity of K over A (expressed in terms of C and N), from A1–A4.

[6] Axiom A5 was shown to be dependent by both Meredith [17] and Chang [5]. Recently, Wos [31, §11.4.3] has used OTTER to find much shorter and more elegant proofs of the dependence of A5 (see footnote 10). Since A5 is dependent, we will restrict ourselves to axioms A1–A4 when working in the C-N fragment of $Ł_{\aleph_0}$. Although A5 is dependent, it plays a crucial role in our proofs of distributivity.

[7] Łukasiewicz and his contemporaries allowed themselves both the rule of detachment and the rule of arbitrary substitution (or instantiation) into theorems. In the late 1950's, Meredith (as explained by Prior [19, Appendix II]) eliminated the arbitrariness of the rule of substitution by insisting that all instantiations be *most general* (much like what we would now call most general *unification*). This was the birth of *condensed* detachment (CD), which is easily implemented in OTTER using hyperresolution. (CD) is the main rule of inference used throughout this paper, and it is what makes the automation of logical calculi feasible [31, page 11]. See [10] for a detailed discussion of the history and automated implementation of condensed detachment, as well as a proof of its equivalence to detachment and substitution.

3.3. Distributivity in $Ł_{\aleph_0}$ I: From the C-N Axioms A1–A4

As we have shown above, it is not difficult to verify that the following distributive law (now expressed in terms of the underlying implication and negation connectives C and N) is valid in the semantics of $Ł_{\aleph_0}$.

(2) $CNCCNpNCCqrrNCCqrrCCNCCNpNqNqNCCNpNrNrNCCNpNrNr$

Therefore, using the completeness result of Rose and Rosser [23], we may infer that a condensed detachment proof of (2) from the axioms A1–A4 must exist. This is one of the (two[8]) distributive laws that Rose and Rosser [23] were unable to prove from the axioms A1–A4 of $Ł_{\aleph_0}$.

Obtaining a pure CD proof of (2) from A1–A4 proved to be quite a challenge. We began by using Otter to find CD proofs of as many as possible of the (fifty or so) lemmas that Rose and Rosser [23, pp. 6-13] prove from A1–A4. We were able to obtain Otter proofs of almost all of Rose and Rosser's lemmas. But, the distributivity laws still remained beyond our reach. At this point, we used a technique developed by Wos and McCune [32] which combines equational reasoning with CD reasoning. We exploited various metatheoretic equalities (and a metatheoretic substitution lemma), which were established by Rose and Rosser.[9] Eventually, this led to a proof of (2) which used both paramodulation and demodulation, as well as CD. This initial, bidirectional, equality/CD proof was then converted, first into a forward, paramodulation/CD proof (with the help of Bob Veroff), and then into a pure CD proof, using an Otter technique developed by Mccune [32, 14] for translating equality substitution steps into CD steps in certain logical calculi, including Łukasiewicz's infinite-valued logic $Ł_{\aleph_0}$.

Our first pure CD proof of (2) from A1–A4 was nearly 100 steps long and made extensive use of double negation terms (*i.e.*, terms of the form 'NNp'). Larry Wos, using a variety of Otter techniques (some of which are discussed in [30]), was eventually able to find an 85 step proof which is free of 'NNp' terms (we call such proofs "term-avoidant"). Here is Wos's term-avoidant Otter proof of (2).[10,11]

1. [CD] $Cpq, p \Rightarrow q$
2. [A1] $CpCqp$
3. [A2] $CCpqCCqrCpr$
4. [A3] $CCCpqqCCqpp$
5. [A4] $CCNpNqCqp$
6. [1,3,3] $CCCCpqCrqsCCrps$
7. [1,3,2] $CCCpqrCqr$
8. [1,3,4] $CCCCpqqrCCCqppr$
9. [1,3,5] $CCCpqrCCNqNpr$
10. [1,6,6] $CCpCqrCCsqCpCsr$
11. [1,6,3] $CCpqCCCprsCCqrs$
12. [1,3,7] $CCCpqrCCCspqr$
13. [1,7,5] $CNpCpq$
14. [1,7,4] $CpCCpqq$
15. [1,12,7] $CCCpCqrsCrs$
16. [1,11,13] $CCCNpqrCCCpsqr$
17. [1,12,14] $CCCpqrCCCqrss$
18. [1,10,14] $CCpCqrCqCpr$
19. [1,3,15] $CCCpqrCCCsCtpqr$
20. [1,16,5] $CCCpqNrCrp$
21. [1,3,18] $CCCpCqrsCCqCprs$
22. [1,18,3] $CCpqCCrpCrq$
23. [1,10,22] $CCpCqrCCrsCpCqs$
24. [1,22,18] $CCpCqCrsCpCrCqs$
25. [1,22,4] $CCpCCqrrCpCCrqq$
26. [1,23,9] $CCpqCCCrspCCNsNrq$
27. [1,24,24] $CCpCqCrsCrCpCqs$
28. [1,24,25] $CCpCCqrrCCrqCpq$
29. [1,27,3] $CpCCpqCCqrr$
30. [1,23,28] $CCCpqrCCpCCqssCCsqr$
31. [1,21,28] $CCCpqCrqCCqpCrp$

[8] The other distributive law that Rose and Rosser were unable to prove from A1–A4 is that disjunction A distributes over conjunction K (*i.e.*, that $CKApqAprApKqr$ follows from A1–A4). With the help of Otter, we have also obtained proofs of this form of distributivity. We omit these proofs both due to space considerations, and because the results reported here are, ultimately, sufficient (given that K and A form a lattice under the ordering imposed by C in $Ł_{\aleph_0}$, see section 4.3 for discussion) to establish both the K-over-A and the A-over-K directions.

[9] For instance, the equalities $NNp = p$ [23, (3.4)] and $Cpq = CNqNp$ [23, (3.5)] are very useful for finding "hybrid" equality/CD proofs of this kind, as are many of the other equalities that are established by Rose and Rosser. It is interesting to note that we have recently used Otter to generate a 19 step paramodulation proof of (2), using only equalities that were proven by Rose and Rosser themselves.

[10] Wos's proof of (2) contains, as a strict sub-proof, a 41 step term-avoidant proof of axiom A5 (line 74 of the proof) from axioms A1–A4. This gives the above proof of (2) the added distinction of containing a radically different proof of the dependence of A5 than is traditionally seen in the literature (*e.g.*, the proofs of Meredith [17] and Chang [5] make extensive use of double-negation). Using Otter, Wos [31, §11.4.3] has discovered a 32 step term-avoidant dependence proof. His Otter proof is significantly more elegant than the proofs previously reported in the literature.

[11] Some technical details: This Otter proof is an 85 step, level 34 proof which uses 6 distinct variables and requires a `max_weight` of 48. If you want to see all the gory details behind our proofs, Otter's `build_proof_object` feature [14] allows the user to see all of the unifications and instantiations involved in any Otter proof.

32. [1,30,20] $CCCpqCCNrssCCsNrCrp$
33. [1,11,31] $CCCCCpqCrqstCCCCqpCrpst$
34. [1,3,31] $CCCCpqCrqsCCCqpCrps$
35. [1,32,5] $CCpNqCqNp$
36. [1,33,31] $CCCCpqCrqCsCrpCCCrpCqpCsCqp$
37. [1,33,24] $CCCCpqCrqCsCtuCCCqpCrpCtCsu$
38. [1,18,35] $CpCCqNpNq$
39. [1,22,36] $CCpCCCqrCsrCtCsqCpCCCsqCrqCtCrq$
40. [1,10,38] $CCpCqNrCrCpNq$
41. [1,39,29] $CCpqCCCrpCqpCCCrqCrpCqp$
42. [1,16,40] $CCCpqCrNsCsCNpNr$
43. [1,40,38] $CpCqNCpNq$
44. [1,6,42] $CCpqCrCNqNp$
45. [1,23,43] $CCNCpNqrCpCqr$
46. [1,9,43] $CCNpNqCrNCCqpNr$
47. [1,28,44] $CCCNpNqrCCqpr$
48. [1,3,45] $CCCpCqrsCCNCpNqrs$
49. [1,11,46] $CCCCNpNqrsCCCtNCCqpNtrs$
50. [1,23,47] $CCpqCCCNrNspCCsrq$
51. [1,8,47] $CCCNpNqNqCCpqNp$
52. [1,3,47] $CCCCpqrsCCCNqNprs$
53. [1,22,48] $CCpCCqCrstCpCCNCqNrst$
54. [1,22,49] $CCpCCCNqNrstCpCCCuNCCrqNust$
55. [1,50,5] $CCCNpNqCNrNsCCqpCsr$
56. [1,7,51] $CNpCCqpNq$
57. [1,34,55] $CCCNpNqCNrNqCCpqCpr$
58. [1,55,35] $CCNpqCNqp$
59. [1,18,56] $CCpqCNqNp$
60. [1,10,56] $CCpCqrCNrCpNq$
61. [1,22,57] $CCpCCNqNrCNsNrCpCCqrCqs$
62. [1,22,58] $CCpCNqrCpCNrq$
63. [1,26,59] $CCCpqCrsCCNqNpCNsNr$
64. [1,59,51] $CNCCpqNpNCCNpNqNq$
65. [1,62,60] $CCpCqrCNCpNqr$
66. [1,61,63] $CCCpqCprCCqpCqr$
67. [1,22,64] $CCpNCCqrNqCpNCCNqNrNr$
68. [1,52,66] $CCCNpNqCqrCCpqCpr$
69. [1,10,66] $CCpCqrCCCrqCrsCpCqs$
70. [1,66,17] $CCpCCqpqCpq$
71. [1,11,67] $CCCCpNCCqrNqstCCCpNCCNqNrNrst$
72. [1,3,68] $CCCCpqCprsCCCNpNqCqrs$
73. [1,69,5] $CCCpqCprCCNpNqCqr$
74. [1,21,70] $CCCpqCqpCqp$

75. [1,22,71] $CCpCCCqNCCrsNrtuCpCCCqNCCNrNsNstu$
76. [1,22,72] $CCpCCCqrCqstCpCCCNqNrCrst$
77. [1,19,74] $CCCpCqCrsCsrCsr$
78. [1,22,76] $CCpCqCCCrsCrtuCpCqCCCNrNsCstu$
79. [1,4,77] $CCCpqCrCsCqpCrCsCqp$
80. [1,79,41] $CCCpqCrqCCCprCpqCrq$
81. [1,76,80] $CCCpqCrqCCCNpNrCrqCrq$
82. [1,54,81] $CCCpqCrqCCCsNCCrpNsCrqCrq$
83. [1,75,82] $CCCpqCrqCCCrNCCNrNpNpCrqCrq$
84. [1,25,83] $CCCpqCrqCCCrqCrNCCNrNpNpCrNCCNrNpNp$
85. [1,37,84] $CCCpqCrqCrCCCrpCrNCCNrNqNqNCCNrNqNq$
86. [1,21,85] $CCpCCqrrCpCCCpqCpNCCNpNrNrNCCNpNrNr$
87. [1,73,86] $CCNpNCCqrrCCCqrrCCCpqCpNCCNpNrNrNCCNpNrNr$
88. [1,78,87] $CCNpNCCqrrCCCqrrCCCNpNqCqNCCNpNrNrNCCNpNrNr$
89. [1,65,88] $CNCCNpNCCqrrNCCqrrCCCNpNqCqNCCNpNrNrNCCNpNrNr$
90. [1,53,89] $CNCCNpNCCqrrNCCqrrCCNCCNpNqNqNCCNpNrNrNCCNpNrNr$

4. An Axiomatic Framework for Positive Sentential Logics

4.1. A Minimal Positive Sentential Logic $\mathbf{M}^+$

We will take as our minimal (non-distributive) positive logic, the following set of two rules and nine axioms. We call this system $\mathbf{M}^+$.[12]

(CD) $Cpq, p \Rightarrow q$

(CA) $p, q \Rightarrow Kpq$

(M1) $CKpqp$

(M2) $CKpqq$

(M3) $CpApq$

(M4) $CqApq$

(M5) $CKCpqCprCpKqr$

(M6) $CKCprCqrCApqr$

(M7) Cpp

(M8) $CCqrCCpqCpr$

(M9) $CCpqCCqrCpr$[13]

[12] $\mathbf{M}^+$ is the same as the positive fragment of Slaney's [25] system **LTW**. In other words, $\mathbf{M}^+$ is just the non-distributive, positive fragment of the system **TW**. The name **TW** traces back to [4]. See, for instance, [20, pp. 59-60] and [21, pp. 39-40] for outlines of the hierarchy of substructural logics (in which $\mathbf{TW}^+$ is near the bottom).

[13] Axiom M9 is A2 from $Ł_{\aleph_0}$. This fact will be used without comment hereafter.

(CA) is called the rule of (condensed) adjunction. Axioms M1–M6 encode basic lattice properties for conjunction (K) and disjunction (A), with respect to the order imposed by the conditional (C). Axioms M7–M9 ensure that the conditional has certain minimal properties, such as reflexivity and transitivity. The axioms and rules of $\mathbf{M}^+$ are valid in a wide class of sentential logics, including the vast majority of substructural logics, as well as intuitionistic logic, $Ł_{\aleph_0}$ (see footnote 16), and classical logic [20, pp. 59-60].

As it stands, $\mathbf{M}^+$ is non-distributive (see section 4.4 for a proof of this claim). That is, the distributivity formula (1) is *not* deducible from the rules and axioms of our minimal system $\mathbf{M}^+$. In the next section, we will present a set of (classical) C-axioms which is sufficient to force distributivity in $\mathbf{M}^+$. We will also provide a (classical) proof of this.

4.2. Three Classical C-Axioms That Make $\mathbf{M}^+$ Distributive

Adding certain sets of implicational axioms to $\mathbf{M}^+$ will force distributivity. One such set consists of the following three classical C-axioms:

(C1) $CpCqp$
(C2) $CCpCqrCqCpr$
(C3) $CCpCpqCpq$

Each of C1–C3 is a theorem in both intuitionistic sentential logic and classical sentential logic. The following proof exemplifies how distributivity is traditionally established, both intuitionistically and classically:

1. [CD] $Cpq, p \Rightarrow q$
2. [CA] $p, q \Rightarrow Kpq$
3. [M1] $CKpqp$
4. [M2] $CKpqq$
5. [M3] $CpApq$
6. [M4] $CqApq$
7. [M5] $CKCpqCprCpKqr$
8. [M6] $CKCprCqrCApqr$
9. [M7] Cpp
10. [M8] $CCqrCCpqCpr$
11. [M9] $CCpqCCqrCpr$
12. [C1] $CpCqp$
13. [C2] $CCpCqrCqCpr$
14. [C3] $CCpCpqCpq$
15. [2,6,5] $KCpApqCrArs$
16. [1,10,7] $CCpKCqrCqsCpCqKrs$
17. [1,10,5] $CCpqCpAqr$
18. [1,11,4] $CCpqCKprq$

19. [1,11,3] $CCpqCKrpq$
20. [1,12,9] $CpCqq$
21. [1,12,6] $CpCqArq$
22. [1,8,15] $CApqAqp$
23. [1,10,17] $CCpCqrCpCqArs$
24. [2,20,12] $KCpCqqCrCsr$
25. [1,13,21] $CpCqArp$
26. [1,10,22] $CCpAqrCpArq$
27. [1,7,24] $CpKCqqCrp$
28. [1,16,27] $CpCqKqp$
29. [1,23,28] $CpCqAKqpr$
30. [2,29,25] $KCpCqAKqprCsCtAus$
31. [1,8,30] $CApqCrAKrpq$
32. [1,19,31] $CKpAqrCsAKsqr$
33. [1,13,32] $CpCKqArsAKprs$
34. [1,18,33] $CKpqCKrAstAKpst$
35. [1,14,34] $CKpAqrAKpqr$
36. [1,26,35] $CKpAqrArKpq$
37. [2,4,36] $KCKpqpCKrAstAtKrs$
38. [1,10,36] $CCpKqArsCpAsKqr$
39. [1,7,37] $CKpAqrKpArKpq$
40. [1,38,39] $CKpAqrAKpqKpr$

This proof makes essential use of C3 (*a.k.a.*, "contraction"), which is not a theorem of $Ł_{\aleph_0}$.[14] In the next section, we will show that adding the three $Ł_{\aleph_0}$ C-axioms A1, A3, and A5 to $\mathbf{M}^+$ also forces distributivity. Not only will this provide a proof that $Ł^+_{\aleph_0}$ (hence, $Ł_{\aleph_0}$) is distributive, it will also provide a novel alternative way to establish distributivity in classical logic (since A1, A3, and A5 are all classical tautologies).[15]

4.3. Distributivity in $Ł_{\aleph_0}$ II: From $\mathbf{M}^+$ to $Ł^+_{\aleph_0}$ with C-Axioms

It is not difficult to show that all of the axioms and rules of our minimal system $\mathbf{M}^+$ are theorems of $Ł_{\aleph_0}$.[16] Moreover, it was shown by Rose [22] that axioms A1, A2, A3, and A5 (plus the rule (CD), of course) form a complete axiomatization for the implicational fragment (*i.e.*, the C fragment) of $Ł_{\aleph_0}$. So, a natural question to ask is whether the system consisting of $\mathbf{M}^+$, plus the implicational axioms A1, A3, and A5 form

[14] For a proof that (1) does not follow from $\mathbf{M}^+$, C1, and C2, see section 4.4. Note: C1 and C2 *are* theorems of $Ł_{\aleph_0}$. C1 is A1, and C2 is theorem (2.6) in [23].

[15] A5 is not intuitionistically valid. So, this proof of (1) is non-intuitionistic.

[16] Specifically, (M1) is (3.13) in Rose and Rosser [23], (M2) is their (3.14), (M3) is (2.4), (M4) is (2.5), (M5) is (3.19) (in axiom form), (M6) is (2.20) (in axiom form), (M7) is (2.10), (M8) is (2.8), and (CA) is (3.22) (in rule form).

a complete axiomatization of the positive fragment (*i.e.*, the *C*-*K*-*A* fragment) of $Ł_{\aleph_0}$. Interestingly, the answer to this question is "yes."[17]

By reasoning in the positive fragment $Ł^+_{\aleph_0}$ of $Ł_{\aleph_0}$, we were able to find a proof of (1). While our substitution and detachment proof of (1) did not directly yield a condensed detachment proof of (1), it did provide us with enough insight to eventually coax OTTER to generate a pure hyperresolution proof of (1). This was achieved using a combination of the resonance and hints strategies developed by Larry Wos [29] and Bob Veroff [27], respectively. The following OTTER proof captures the essence of our original hand proof of (1) in $Ł^+_{\aleph_0}$.

1. [CD] $Cpq, p \Rightarrow q$
2. [CA] $p, q \Rightarrow Kpq$
3. [M1] $CKpqp$
4. [M2] $CKpqq$
5. [M3] $CpApq$
6. [M4] $CqApq$
7. [M5] $CKCpqCprCpKqr$
8. [M6] $CKCprCqrCApqr$
9. [M7] Cpp
10. [M8] $CCqrCCpqCpr$
11. [M9] $CCpqCCqrCpr$
12. [A1] $CpCqp$
13. [A3] $CCCpqqCCqpp$
14. [A5] $CCCpqCqpCqp$
15. [1,10,8] $CCpKCqrCsrCpCAqsr$
16. [1,10,7] $CCpKCqrCqsCpCqKrs$
17. [1,10,6] $CCpqCpArq$
18. [1,10,5] $CCpqCpAqr$
19. [1,11,11] $CCCCpqCrqsCCrps$
20. [1,11,5] $CCApqrCpr$
21. [1,11,4] $CCpqCKrpq$
22. [1,11,12] $CCCpqrCqr$
23. [1,12,3] $CpCKqrq$
24. [1,12,9] $CpCqq$
25. [1,19,19] $CCpCqrCCsqCpCsr$
26. [1,11,20] $CCCpqrCCApsqr$

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[17] With some care, the completeness of $\mathbf{M}^+$, plus A1, A3, and A5 for $Ł^+_{\aleph_0}$ can be established using [24, Theorem 4.1, p. 434]. Some authors (*e.g.*, [26]) have interpreted Scott's [24] results as showing that $Ł^+_{\aleph_0}$ is sound and complete with respect to abelian ordered groups. This is not correct. *Pace* Urquhart [26, p. 104], Axiom A3 of $Ł^+_{\aleph_0}$ is *not* valid in such structures. What Scott *proves* is that $Ł^+_{\aleph_0}$ is sound and complete with respect to *the non-negative fragment* of abelian ordered groups. So, the statement of Scott's soundness theorem [24, Theorem 3.1, p. 424] is misleading.

27. [1,22,13] $CpCCpqq$
28. [2,23,21] $KCpCKqrqCCstCKust$
29. [2,9,24] $KCppCqCrr$
30. [2,24,9] $KCpCqqCrr$
31. [1,25,27] $CCpCqrCqCpr$
32. [1,7,28] $CCpqKCKrsrCKtpq$
33. [1,7,29] $CpKpCqq$
34. [1,7,30] $CpKCqqp$
35. [1,31,13] $CCpqCCCqppq$
36. [1,16,32] $CCpqCKrpKrq$
37. [1,15,33] $CCpqCApqq$
38. [1,15,34] $CCpqCAqpq$
39. [1,25,35] $CCpCCqrrCCrqCpq$
40. [1,10,36] $CCpCqrCpCKsqKsr$
41. [1,11,37] $CCCApqqrCCpqr$
42. [1,39,26] $CCpAqrCCCqppAqr$
43. [1,40,38] $CCpqCKrAqpKrq$
44. [1,41,21] $CCpqCKrApqq$
45. [1,42,6] $CCCpqqApq$
46. [1,11,43] $CCCKpAqrKpqsCCrqs$
47. [1,11,44] $CCCKpAqrrsCCqrs$
48. [1,45,14] $ACpqCqp$
49. [1,46,18] $CCpqCKrAqpAKrqs$
50. [1,47,17] $CCpqCKrApqAsq$
51. [2,50,49] $KCCpqCKrApqAsqCCtuCKvAutAKvuw$
52. [1,8,51] $CACpqCqpCKrApqAKrpq$
53. [1,52,48] $CKpAqrAKpqr$
54. [2,6,5] $KCpAqpCrArs$
55. [1,8,54] $CApqAqp$
56. [1,10,55] $CCpAqrCpArq$
57. [1,56,53] $CKpAqrArKpq$
58. [2,3,57] $KCKpqpCKrAstAtKrs$
59. [1,10,57] $CCpKqArsCpAsKqr$
60. [1,7,58] $CKpAqrKpArKpq$
61. [1,59,60] $CKpAqrAKpqKpr$

We find this proof more intuitive and explanatory than the *C*-*N* fragment proof reported in section 3.3. The most important step of this high-level proof is showing that, once axioms A1, A3, and A5 are added to $\mathbf{M}^+$, Apq is *provably equivalent to* $CCpqq$ (step 45). Once this equivalence is established, the following important *linearity* formula follows immediately, by axiom A5 (step 48).

(3) $ACpqCqp$

It follows from (3) that the lattice ordering imposed by C in $Ł^{+}_{\aleph_0}$ is *total*. It is well known that all totally ordered lattices are distributive.

4.4. Non-Distributivity in Some Substructural Logics

We have already seen two sets of C axioms which force the distributivity of K over A (and *vice versa*) in $\mathbf{M}^{+}$. It is well known that many substructural extensions of $\mathbf{M}^{+}$ are *non*-distributive [3]. However, as far as we know, the non-distributivity of many of these substructural logics has never been established (in the literature) by the presentation of *logical matrices* in which all the axioms and rules of the logic are satisfied, but distributivity is violated. For instance, the relevance logic $\mathbf{R}^{+}$ (*sans* distributivity axiom), which is given by $\mathbf{M}^{+}$, plus the following two additional C axioms:

(R1) $CCpCpqCpq$

(R2) $CCpCqrCqCpr$

is known to be non-distributive, but we have never seen matrices which establish this fact about $\mathbf{R}^{+}$.

The $M_3 - N_5$ theorem from lattice theory [8, page 134] tells us that the smallest non-distributive lattices are of order 5. So, the smallest matrices which violate (1), but obey the lattice properties implicit in $\mathbf{M}^{+}$ will contain at least five elements. As a result, using a general purpose model finding program such as Mace [15] to look for matrices of the kind we need is not feasible, unless one can find a way to prune the search space considerably. The $M_3 - N_5$ theorem provides exactly the kind of information needed to shrink the search space. By exploiting the $M_3 - N_5$ theorem, we were able to create Mace input files that narrow the space of 5 element models to a very small number of possible candidates which are either of the M_3 or the N_5 structure. This enabled us to find the following 5 element matrices, thus establishing the non-distributivity of $\mathbf{R}^{+}$.[18,19]

[18] There are two designated values for these matrices: *1* and *4* (indicated by *).

[19] Of course, these matrices also suffice to show that $\mathbf{M}^{+}$ is non-distributive. And, since $\mathbf{E}^{+}$ (the positive fragment of Anderson and Belnap's [1] system $\mathbf{E}$) is strictly weaker than $\mathbf{R}^{+}$, they also establish that $\mathbf{E}^{+}$ is non-distributive. Moreover, our matrices also satisfy the '*mingle*' axiom $CpCpp$, thus establishing that the system $\mathbf{RM}^{+}$ is not distributive. Finally, these matrices also satisfy the *linearity* or *connectedness* condition (3). Therefore, adding linearity to $\mathbf{RM}^{+}$ doesn't yield distributivity either. See [9] for a detailed discussion of $\mathbf{E}$, $\mathbf{R}$, $\mathbf{RM}$, and other relevant sentential logics.

C	0	1	2	3	4
0	1	1	1	1	1
*1	0	1	2	3	2
2	3	1	1	3	1
3	2	1	2	1	2
*4	0	1	2	3	4

K	0	1	2	3	4
0	0	0	0	0	0
*1	0	1	2	3	4
2	0	2	2	0	2
3	0	3	0	3	0
*4	0	4	2	0	4

A	0	1	2	3	4
0	0	1	2	3	4
*1	1	1	1	1	1
2	2	1	2	1	4
3	3	1	1	3	1
*4	4	1	4	1	4

Matrices which Establish the Non-Distributivity of $\mathbf{R}^{+}$

A more difficult example concerns the logic $\mathbf{BCK}^{+}$, which is given by $\mathbf{M}^{+}$, plus the following two additional C axioms.[20]

(BCK1) $CpCqp$

(BCK2) $CCpCqrCqCpr$

$\mathbf{BCK}^{+}$ is shown in [18] to be non-distributive by a complicated metatheoretic argument to the effect that the distributivity formula (1) cannot be derived in a (cut-free) Gentzen system for $\mathbf{BCK}^{+}$. Using our $M_3{-}N_5$ Mace technique, we were able to show that there are no 5-element matrices in which all the axioms and rules of $\mathbf{BCK}^{+}$ are satisfied but the distributivity formula (1) is violated. For several weeks, we tried to extend our Mace technique to spaces with dimension greater than five, to no avail. The search space just got too big, and our $M_3 - N_5$ tricks were of little help in making the space manageable for Mace searches.

Eventually, we stumbled upon John Slaney's program MaGIC [25], which is a model finding program designed and optimized specifically for substructural logics of the kind we were studying. Amazingly, in just a few seconds, MaGIC found the following 6 element matrices for C, K, and A which establish the non-distributivity of $\mathbf{BCK}^{+}$.[21]

C	0	1	2	3	4	5
0	5	5	5	5	5	5
1	4	5	4	4	5	5
2	4	4	5	4	5	5
3	4	4	4	5	5	5
4	4	4	4	4	5	5
*5	0	1	2	3	4	5

K	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	0	0	1	1
2	0	0	2	0	2	2
3	0	0	0	3	3	3
4	0	1	2	3	4	4
*5	0	1	2	3	4	5

A	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	1	4	4	4	5
2	2	4	2	4	4	5
3	3	4	4	3	4	5
4	4	4	4	4	4	5
*5	5	5	5	5	5	5

Matrices which Establish the Non-Distributivity of $\mathbf{BCK}^{+}$

[20] Note: axiom BCK1 is none other than axiom A1 from $Ł_{\aleph_0}$, *i.e.*, formula C1 (*a.k.a.*, "weakening"), and BCK2 is none other than the formula C2 (*a.k.a.*, "permutation"). Recall, both BCK1 and BCK2 were used in our distributivity proof in section 4.2. So, the fact that $\mathbf{BCK}^{+}$ lacks "contraction" (*viz.*, C3) is crucial here.

[21] The only designated value in these matrices is *5* (indicated by *).

We end this section by noting that non-distributive matrices of dimension less than 9 exist for all fragments of $Ł^+_{\aleph_0}$, except those containing *both* axiom A1 *and* axiom A3 (over and above $\mathbf{M}^+$). This raises an interesting and challenging open question: Is axiom A5 *necessary* for the distributivity of $Ł^+_{\aleph_0}$? That is, can distributivity be deduced from $\mathbf{M}^+$, A1, and A3 alone? And, if not, are there finite logical matrices which establish that (1) does not follow from $\mathbf{M}^+$, A1, and A3?

5. Conclusion: Some Lessons Learned

Our investigations into distributivity in positive sentential logics were greatly enhanced by the automated reasoning programs OTTER, MACE, and MAGIC. Along the way, we have learned a couple of important lessons about automated reasoning in sentential logics:

- OTTER (and other, general-purpose resolution theorem provers) is very good at reasoning in systems with few rules, few axioms, and few connectives. But, its effectiveness seems to decrease markedly as the number of rules, axioms, and connectives increases. In the underlying *C*-*N* fragment of $Ł_{\aleph_0}$, which has only 1 rule, 4 axioms, and 2 connectives, OTTER was able to find proofs of many important lemmas, and to fill-in some large gaps in our "hybrid" equality/CD proof sketches of various distributivity theorems. In fact, we are now able to get OTTER to prove the *C*-*N* rendition of distributivity (2) on its own, without any guidance. Indeed, OTTER can generate many novel and elegant proofs of (2). On the other hand, in the high-level *C*-*K*-*A* fragment of $Ł_{\aleph_0}$, which has 2 rules, 9 axioms, and 3 connectives, OTTER seems to get bogged down in the syntactic complexity, thus requiring substantially more guidance to obtain the desired theorems. It is interesting that we humans seem to be better at reasoning in systems with more connectives, axioms, and rules (which tends to be correlated with *shorter formulas* in proofs, as is evidenced by our proofs of (1) and (2)); whereas, OTTER seems to thrive on a minimum of connectives, rules and axioms (*independently* of how *long* the formulas get). We would like to see strategies developed to improve OTTER's performance in axiomatic systems with numerous rules, axioms, and connectives.
- When searching for matrices to establish independence results in sentential logics (as we have done here for distributivity in substructural logics), it is often *necessary* (because of the massive and exponentially increasing search spaces) to use some technique for pruning or optimizing the search space. We were able to exploit the $M_3 - N_5$ theorem from lattice theory to greatly reduce the number of candidate matrices in our MACE searches for non-distributive lattices. However, when it came to lattices of order greater than 5, our $M_3 - N_5$ tricks were of little help, and we were forced to use MAGIC, which has built-in optimizations for the class of substructural logics we were investigating. *Special purpose* model finders like MAGIC are of great value in problems of this kind.

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