Closure Under Conditionalization

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- What is Closure Under Conditionalization (CUC)?
  - *Via* (CUC)'s role in Lewis's argument against **The Equation**
- Four other historical applications of (CUC)
  - (CUC) and Bradley's **Preservation** principle [1]
  - (CUC) and Russell & Hawthorne's **Probably** [8]

  - (CUC) and Leitgeb's **Lockean Thesis** [5]
  - (CUC) and Lewis's **Principal Principle** [7]
- The first three trivialities can be blocked by a simple quantifier restriction strategy, applied to (CUC). The last two cases, however, are not as easily handled.

Consider the following constraint [11] on a probability function $\Pr(\cdot)$, over a language containing the three atoms $P \to Q$, P, and Q (where the atom $P \to Q$ is extra-systematically interpreted as the indicative conditional asserting that if P, then Q).

The Equation. $\Pr(P \to Q) = \Pr(Q \mid P)$.

Lewis [6] assumes that the set $\mathscr{P}$ of probability functions that satisfies **The Equation** is *closed under conditionalization*.

As he points out, this assumption is equivalent to the following strengthening of **The Equation**.

The Resilient Equation.[1] For all propositions x such that $\Pr(P \,\&\, x) > 0$, $\Pr(P \to Q \mid x) = \Pr(Q \mid P \,\&\, x)$.

Various triviality results have been derived from **The Resilient Equation.** The strongest possible such triviality result [4] is

$$\Pr(P \,\&\, (Q \equiv (P \to Q))) = 1.$$

[1] I borrow the term **Resilient** from Skyrms [10].

More generally, suppose the set of probability functions $\mathscr{P}_C$ satisfying constraint C is closed under conditionalization.

Then, for any member $\Pr(\cdot)$ of $\mathscr{P}_C$, C_x holds for all propositions x for which C_x is well-defined. Where, C_x is obtained by conditionalizing all occurrences of $\Pr(\cdot)$ in C on x.

In Lewis's case, C^{Eq} is **The Equation.** The claim that C_x^{Eq} holds for all x such that C_x^{Eq} is well-defined is **The Resilient Equation**.

Generally, call C **The Constraint** and $(\forall x)C_x$ **The Resilient Constraint**. In general, **The Resilient Constraint** will be *much stronger than* **The Constraint**. And, often, **TR**C will be *trivial*.

Lewis's argument against **The Equation** is, ultimately, a *normative* (or *evaluative* one). Next, I will try to get clearer on the deontic-logical structure of Lewis's (CUC) assumption.

Then, I'll propose a simple quantifier restriction strategy for avoiding Lewisian triviality. This strategy will be effective when applied to some (but not all) other (CUC)-generated trivialities.

(CUC^0_C) If C is *rationally permissible*, then so is $(\forall x)C_x$.

(CUC$_C$) If C is a *rational requirement*, then so is $(\forall x)C_x$.

(CUC^0_C) fails for *many* constraints C. To wit:

Let $\Pr(\cdot)$ be the credence function of an agent S (at time t), and let P be some (particular) proposition such that the following constraint is rationally permissible for S (at t).

(C) $1 > \Pr(P) > 0$.

Suppose (for *reductio*) that $(\forall x)C_x$ is permissible (for S at t), for all x such that C_x is well-defined. Then, we should have $\Pr(P \mid \neg P) > 0$. But, this is *probabilistically incoherent*.

☞ In fact, as we'll see, for *any* combination of deontic operators, there are conditions C for which the corresponding (CUC) fails.

For now, I will plump for (CUC$_{C^{\text{Eq}}}$) in my reconstruction of Lewis's argument. I will return to this question in the Epilogue.

My reconstruction: Lewis assumed (for *reductio*) that **The Equation** is a rational requirement. Then, he (and others) used (CUC$_{C^{\text{Eq}}}$) to complete his (their) *reductio*(s) of **The Equation.**

(1) **The Equation** is a rational requirement (for *reductio*).
(2) (CUC$_{C^{\text{Eq}}}$)
(3) ∴ **The Resilient Equation** is a rational requirement.
(4) But, $\Pr(P \,\&\, (Q \equiv (P \to Q))) = 1$ is *not* a rational requirement.
(5) Contradiction.

Why interpret Lewis's argument as a *reductio* of **The Equation** — *as opposed to a reductio of* (CUC$_{C^{\text{Eq}}}$)? After all, as Lewis was well aware, there are constraints C which *violate* (CUC$_C$).

Next, I will discuss four other historical applications of (CUC).

The first two will be similar to Lewis's, as they: (a) involve conditionals and/or modals, and (b) can be parried by a simple quantifier restriction strategy. The last two will be different.

Richard Bradley [1] endorses the following rational requirement (assuming the same conventions I used in my discussion of Lewis above).

Preservation. $\Pr(\cdot)$ satisfies the following constraint:

(C^B) If $\Pr(P) > 0$ and $\Pr(Q) = 0$, then $\Pr(P \to Q) = 0$.
If $\Pr(P) > 0$ and $\Pr(Q) = 1$, then $\Pr(P \to Q) = 1$.

Preservation is considerably weaker than **The Equation** (*e.g.*, it imposes *no* constraint on regular agents). However, when we apply (CUC) to **Preservation**, we get **Resilient Preservation.**

Resilient Preservation. $\Pr(\cdot)$ satisfies the following constraint, for all x such that C^B_x is well-defined:

(C^B_x) If $\Pr(P \mid x) > 0$ and $\Pr(Q \mid x) = 0$, then $\Pr(P \to Q \mid x) = 0$.
If $\Pr(P \mid x) > 0$ and $\Pr(Q \mid x) = 1$, then $\Pr(P \to Q \mid x) = 1$.

Resilient Preservation yields its own Lewisian trivialities. While these are weaker than Lewis's, they are strong enough to suggest that **Resilient Preservation** is not a rational requirement.

The following table illustrates the differences between **The Equation**, **The Resilient Equation**, and **Resilient Preservation.**

P	Q	$P \to Q$	$\Pr(\cdot)$	$\Pr(\cdot)$ + **The Equation**	$\Pr(\cdot)$ + **R. Preservation**	$\Pr(\cdot)$ + **R. Equation**
T	T	T	a	a	a	a
T	T	F	b	b	0	0
T	F	T	c	c	0	0
T	F	F	d	d	d	$1-a$
F	T	T	e	e	e	0
F	T	F	f	f	0	0
F	F	T	g	$\frac{a+b}{a+b+c+d} - a - c - e$	0	0
F	F	F	$1-\sum$	$1-\sum$	$1-\sum$	0

The Equation reduces the number of $\Pr(\cdot)$'s degrees of freedom by 1 (from 7 to 6), **Resilient Preservation** reduces it by 4, and **The Resilient Equation** reduces it by 6. Moreover, **The Resilient Equation** is *strictly stronger* than **Resilient Preservation.**

Russell & Hawthorne [8] discuss various triviality results. One of these (adapted to our present framework) involves:

Probably. Let P be extra-systematically interpreted as encoding the claim that S is rationally required to assign Q at least $1/2$ credence (at t), and let $\Pr(\cdot)$ be S's credence function at t [where $\Pr(P \,\&\, \neg Q) > 0$]. Then,

($C^{R\&H}$) $\Pr(Q \mid P) \geq 1/2$.

Probably seems like a plausible (candidate for a) rational requirement [2]. What happens if we apply (CUC) to it?

Resilient Probably. Let P assert that S is rationally required to assign Q at least $1/2$ credence (at t), and let $\Pr(\cdot)$ be S's credence function at t [where $\Pr(P \,\&\, \neg Q) > 0$]. Then, for all x such that $C_x^{R\&H}$ is well-defined,

($C_x^{R\&H}$) $\Pr(Q \mid P \,\&\, x) \geq 1/2$.

Resilient Probably is *probabilistically incoherent* (for *any* such S).

These first three triviality arguments are similar in that: (a) they all involve conditionals and/or modals, and (b) they can all be parried by the following simple quantifier restriction strategy.

(CUC$_C^*$) If C is a *rational requirement*, then so is $(\forall x)C_x$, *provided that* $(\forall x)$ *ranges only over* x*'s which satisfy*:

($*$) The probability calculus and C_x do not jointly determine any specific numerical values for $\Pr(\cdot)$.

To see why (CUC$_C^*$) allows conditions (C^{Eq}), (C^B), and ($C^{R\&H}$) to avoid Lewisian triviality arguments, note that:

- $(\forall x)C_x^{\text{Eq}}$ only yields triviality because its instances $C_{\neg Q}^{\text{Eq}}$ and $C_{P \supset Q}^{\text{Eq}}$ entail specific (extremal) numerical values for $\Pr(\cdot)$.
- $(\forall x)C_x^B$ only yields triviality because its instances C_Q^B and $C_{\neg Q}^B$ entail specific (extremal) numerical values for $\Pr(\cdot)$.
- $(\forall x)C_x^{R\&H}$ only yields triviality because its instance $C_{\neg Q}^{R\&H}$ entails a specific (extremal) numerical value for $\Pr(\cdot)$.

Leitgeb [5] proposes a "stability theory of belief," which can be (partially) emulated by applying (CUC*) to **The Lockean Thesis**.

The Lockean Thesis. Let $\Pr(\cdot)$ be S's credence function (at some time t), and let p be any proposition that is believed by S (at t). Then, (C^L) $\Pr(p) > 1/2$.

Leitgeb assumes that (C^L) is a rational requirement (for any S who believes any p at t). Moreover, he assumes (CUC$_{C^L}^*$).

To be more precise, (CUC$_{C^L}^*$) has the effect of strengthening **The Lockean Thesis** — yielding **The Resilient Lockean Thesis**.

The Resilient Lockean Thesis. Let $\Pr(\cdot)$ be S's credence function (at t), and p be any proposition believed by S (at t). Then, for all x s.t. (C_x^L) is well-defined *and* ($*$) the probability calculus + (C_x^L) do not entail any specific numerical values for $\Pr(\cdot)$, (C_x^L) $\Pr(p \mid x) > 1/2$.

The Resilient Lockean Thesis entails that S's belief set (at t) must be deductively cogent [5]. Is this a feature or a bug?

According to (veritistic) EUT, it is a rational requirement that S's belief set (at t) maximize Pr-expected accuracy (MEA).

As Dorst [3] shows, (MEA) implies that violations of **The Lockean Thesis** are rationally prohibited.

Indeed, (MEA) implies something stronger than this. Consider an agent S facing (at t) a large lottery with n tickets.

Assuming the Principle Principal (and sufficiently large n), (MEA) will require S to have an inconsistent belief set (at t).

Thus, we have a conflict between the following three claims:

- The Principal Principle is a rational requirement.
- (MEA) is a rational requirement.
- (CUC$_{C^L}^*$)

This does not involve modals/conditionals (lotteries are factual), and it requires the full strength of (CUC$_{C^L}^*$). See [9] for various other puzzling consequences of Leitgeb's account.

According to Lewis [7], the Principal Principle (PP) is a rational requirement, but (CUC) *cannot* be applied to it *unrestrictedly*.

(PP) $\Pr(p \mid \mathrm{Ch}(p) = c) = c$.

Consider (PP$_x$)

(PP$_x$) $\Pr(p \mid \mathrm{Ch}(p) = c \,\&\, x) = c$.

(CUC) naïvely applied to (PP) would yield $(\forall x)\mathrm{PP}_x$, which asserts that (PP$_x$) holds *for all* x *such that* (PP$_x$) *is well-defined*. But, *that* principle is *probabilistically incoherent* ($\mathrm{Ch}(p) < 1$, $x := p$).

Lewis argues that we only get a rational requirement here *if we restrict the domain of the quantifier* $(\forall x)$ to *admissible evidence*, which *rules-out* conditionalizing on various x's, *e.g.*, $x := p$.

The simple restriction strategy (CUC$^*_{\mathrm{PP}}$) will not suffice here, since some cases in which the probability calculus + (PP$_x$) do *not* entail specific values for $\Pr(\cdot)$ will (nonetheless) be *inadmissible*.

Let's return to Lewis's original application of (CUC) to **The Equation.** Did he give an *argument* for (CUC$_{C^{\mathrm{Eq}}}$)? He says:

> the . . . class of all those probability functions that represent possible systems of beliefs . . . is closed under conditionalizing. Rational change of belief never can take anyone to a subjective probability function outside the class; and . . . the change of belief that results from coming to know an item of new evidence should take place by conditionalizing on what was learned.

Here, Lewis seems to be arguing that the class of rational (*viz.*, rationally permissible?) credence functions is closed under conditioning. Perhaps, but *how is that dialectically relevant*?

What needs to be established here is (II), not merely (I):

(I) If $\Pr(\cdot)$ is rational (*viz.*, permissible/required), then so is $\Pr(\cdot \mid x)$, for any x such that $\Pr(\cdot \mid x)$ is well-defined.

(II) If $\Pr(\cdot)$'s *satisfying* **The Equation** is rational, then so is $\Pr(\cdot)$'s *satisfying* **The Resilient Equation.**

Establishing the inconsistency of (PP), (MEA), and (CUC$^*_{CL}$).

(1) Lottery chances + (PP) ⇒ Lottery Credences $\Pr(\cdot)$.

(2) $\Pr(\cdot)$ + (MEA) ⇒ (Inconsistent) Lottery Beliefs **B** + (**TLT**).

(3) (**TLT**) + (CUC$^*_{CL}$) ⇒ **The Resilient Lockean Thesis.**

(4) **The Resilient Lockean Thesis** ⇒ Consistency of **B**.

(5) Contradiction.

Conjecture. Reformulating Bradley's **Preservation** and **Resilient Preservation** in terms of *Jeffrey conditioning* (and closure under all well-defined Jeffrey-shifts) will *not* yield triviality.

Jeffrey Preservation.

(C^{B^*}) Suppose, at t_0, $\Pr_0(P) > 0$. If S learns (Jeffrey-style, between t_0 and t_1) that $\Pr(Q) \geq \alpha \in (0,1)$, then $\Pr_1(P \to Q) \geq \alpha$.

[1] R. Bradley, *A Preservation Condition for Conditionals*, 2000.

[2] K. Dorst, *Evidence: A Guide for the Uncertain*, to appear.

[3] ______, *Lockeans Maximize Expected Accuracy*, 2017.

[4] B. Fitelson, *The Strongest Possible Lewisian Triviality Result*, 2015.

[5] H. Leitgeb, *The Stability Theory of Belief*, 2016.

[6] D. Lewis, *Probabilities of Conditionals and Conditional Probabilities*, 1976.

[7] ______, *A Subjectivist's Guide to Objective Chance*, 1980.

[8] J.S. Russell and J. Hawthorne, *General Dynamic Triviality Theorems*, 2016.

[9] G. Schurz, *Impossibility Results for Rational Belief*, 2017.

[10] B. Skyrms, *Resiliency, Propensities, and Causal Necessity*, 1977.

[11] R. Stalnaker, *A Theory of Conditionals*, 1968.