DEGREE OF FACTUAL SUPPORT

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We wish to give a precise formulation of the intuitive concept: The degree to which the known facts (the evidence) support a given hypothesis.

Perhaps the best way to clarify exactly which concept we have in mind is to relate it to another concept, the concept of degree of confirmation. This is the measure of the degree of belief a rational being would assign to a hypothesis on the given evidence. (Hence Russell's term 'degree of credibility'[2] seems to be more appropriate than 'degree of confirmation.') Let us use 'F' as an abbreviation of our concept, and 'C' for 'degree of confirmation (credibility).' Carnap[3] has described C as a measure of evidential support, and this description is fitting for both concepts. But C is the logical basis for induction, so we could describe it as the inductive support given by the evidence. As Carnap points out in section 46 (loc. cit.) this support depends on several factors. The most immediate factor is that of the factual support given by the evidence, our F. (Others are the variety of the evidence, its reliability, the simplicity of the hypothesis, etc.) In this paper we analyze this single factor. We could describe F as the degree to which the evidence supports the hypothesis, without any inductive considerations. Its importance lies in that it must be clarified (along with the other factors) before we can define C. In a given inductive judgment we must first see how favorable the facts are; some facts will support our hypothesis, while some will weigh against it, and it is our task to evaluate this—which is accomplished by F. Then we must take into account other factors (like the ones mentioned above) and carry out an induction, resulting in a C-value.

A few remarks are in order to clarify the intuitive idea we start from. A scientist uses this concept whenever he counts the number of observations favorable to his hypothesis, and compares them to the unfavorable observations. E.g., let his hypothesis be that all 150 million Americans are under 7 feet tall, unless they have a glandular disturbance. Then he can count all Americans who were observed to be under 7 feet tall, or were observed to have a glandular disturbance (or both) as favorable evidence; while, e.g., any report of an American over 7 feet tall, that mentioned nothing about his glandular condition, would be unfavorable. The degree of factual support weighs the favorable facts against the unfavorable ones, and weighs the result against the hypothesis. Of course, one single case of an American having no glandular disturbance, but being over 7 feet tall, would completely invalidate the hypothesis, and hence the rest of the evidence would be irrelevant. Equally ir-

[1] In the main the contribution of Paul Oppenheim was limited to propounding some of the fundamental ideas.

[2] Russell, Bertrand, *Human Knowledge*, New York: Simon and Schuster, 1948.

[3] Carnap, Rudolf, *Logical Foundations of Probability*, Chicago: The University of Chicago Press, 1950.

relevant (for *factual* support) would be all evidence as to the height and gland-condition of Europeans (although this would be quite relevant from the *inductive* standpoint).

Our task is the rational reconstruction (or in Carnap's terms *explication*) of the intuitive concept of *degree of factual support*. The commonest procedure of explication is to apply a trial and error method till one arrives at an ingenious guess, and then try to find intuitive reasons to justify the proposed *explicatum*. This procedure is clearly very dangerous: The intuition of the most honest and well-trained philosopher is likely at times to become a tool for grinding an ax. Even Carnap, who begins (loc. cit.) by setting down strict conditions of adequacy for his explicatum and arrives at the class of symmetric C-functions, finally just selects one of these, C^*, and justifies the choice by showing that C^* has satisfactory consequences. We feel that we must first put down clearly all that our intuition tells us about the *explicandum*, and then find the precise definitions that satisfy our intuitive requirements. In this sense we hope to set a standard for explications.

On the basis of this methodology we divide our work into four parts:

1. Purely formal conditions are set down, many of which are conventions; they are useful, but they do not essentially restrict the choice of the explicatum. In this part we determine what kind of a definition we are looking for.
2. We now put down as strictly as possible all that out intuition tells us about F. These material conditions, as well as the formal ones of part 1, will be conditions of adequacy for the explicatum.
3. We characterize the class of definitions which satisfy our conditions of adequacy. We should arrive either at a unique explicatum, or at a small class all of whose members are demonstrably equally well in agreement with intuition.
4. We work out some numerical examples on the basis of the (unique or several) remaining explicata, and prove a few theorems.

1. Formal Conditions of Adequacy. 'Degree of factual support,' as the name suggests, is a functor. It measures a relation between a hypothesis and its evidence. Specifically we will consider the hypothesis and the evidence as stated in a formalized language. In order to avoid great technical difficulties, we will be satisfied to choose a first order functional calculus as our language. In this language we have individual constants and variables (denoting physical objects) and predicates (denoting their properties and relations). There are the logical combinations customary for the functional calculi, and there are no restrictions on the choice of predicates—as long as they are finite in number. This last point is very important; the fact that we allow dependencies between our atomic sentences allows us to use a much richer language than the one usually considered in such papers. E.g., Carnap (loc. cit.) requires independence, and Helmer and Oppenheim[4] also require that we use only one-place predicates. If there are

[4] Helmer, Olaf and Oppenheim, Paul. "A Syntactical Definition of Probability and of Degree of Confirmation," *The Journal of Symbolic Logic*, X, 1945, pp. 25–60.

dependencies, these must be indicated in the form of postulates added to the usual axioms. (E.g., if we want to allow 'male' and 'father' (i.e., male parent) as predicates, we must add the postulate: $(x)(\text{Father}(x) \supset \text{Male}(x))$.) The meaningful statements of this language, i.e., the statements formed according to its rules of syntax, are as usual called 'well-formed-formulas' (abbreviate 'w.f.f.').

Thus F is a purely logical function of two w.f.f., H and E (representing the hypothesis and the evidence respectively). We can now enumerate some conditions of adequacy for our functor:

CA1. $F(H, E)$ is a functor whose two arguments are w.f.f. chosen from our first order functional calculus.[5]

The range of values of F is purely conventional. The following is a very useful convention for this:

CA2. F ranges from -1 to 1, with larger values of F indicating more support. -1 indicates the worst possible support, when the evidence shows that the hypothesis is false; 0 corresponds to evidence that is equally favorable and unfavorable (including the case of irrelevance); 1 indicates the best possible support, when the evidence proves that the hypothesis is true.

The more E supports H, the less it supports its negation $\bar{H}$. We see from CA2 that they get equal support just when F is 0, while when $F(H, E)$ is 1 (meaning that H is proved, hence $\bar{H}$ disproved) $F(\bar{H}, E)$ must be -1. The formal condition which most naturally sums up these points is:

CA3. $$F(\bar{H}, E) = -F(H, E).$$

In view of the intended meaning we must obviously have:

CA4. If $\vdash H \equiv H'$ and $\vdash E \equiv E'$, then $F(H, E) = F(H', E')$.[6]

Our next problem is to assign numbers to H and E so that we can construct the functor F. Equivalent w.f.f. must get equal numbers, and these numbers must somehow correspond to "how much H (or E) expresses." The standard way to introduce such a syntactic measure for w.f.f. is by means of a normal form. In order to save space, we will refer the reader to a text on deductive logic, or the excellent treatment of this by Carnap (loc. cit.) or the similar approach of Helmer and Oppenheim (loc. cit.). We will simply outline the results, and indicate what changes must be made to adapt it to our approach.

We will assume for the moment that there are a finite number, n, of things in the world, i.e., that our universe of discourse has n elements. The following discussion is applicable only for a fixed value of n. Let us give a name to each of these n things in our meta-language, and let us call an *atomic statement* a statement that asserts that one of these things has a certain property, or that several things satisfy a relation; then all possible statements of our language can be represented as combinations of atomic statements. A "possible world"

[5] This condition will be somewhat restricted later, see Part 2.

[6] '$\vdash W$' expresses that W is a theorem, i.e., that it is an analytic consequence of our dependency-postulates, or simply analytic if there are no dependencies.

is as completely described as our language allows if we form a conjunction containing for each atomic statement either it or its negation. Such a conjunction is called by Carnap a *state-description* (abbreviated 's.d.'). Carnap formulates his s.d. in the object-language by requiring that all individuals have names, but this is not at all necessary. Since an s.d. describes a model world completely, every w.f.f. is either true or false *in* an s.d. Carnap shows that the meaning of a w.f.f. is expressed by the disjunction of those s.d. in which it is true. (See section 18, loc. cit.) Then Carnap introduces his regular m-functions, which assign weights to all the s.d., the sum of the weights being 1. The measure of a w.f.f. is the sum of the weights of those s.d. in which it is true. (See section 55, loc. cit.) This has to be modified slightly if there are dependencies. It is easy to see that the effect of having dependencies is that some of our s.d. contradict our postulates. So let us denote by 's.d.' only those maximal conjunctions which are consistent with our postulates. If we interpret 's.d.' in this sense, all that we said above remains true for languages with dependencies.

CA5. For fixed n, we must assign to each s.d. a *weight*, i.e., a non-negative number, the sum of the weights being 1. Then for any w.f.f. A, $m(A)$ is the sum of the weights of those s.d. in which A is true.

Such m-functions play a fundamental role in many explications. We will use an m-function to assign numerical values to the relevant combinations of H and E, and—since different relations between H and E can be expressed as relations between these numbers—they will enable us to translate all relevant information about H and E into numerical terms.

We can thus define m-functions, but in general a given definition will yield different values for the same w.f.f. for different values of n. We do not as a matter of fact know what the actual value of n is (we are not even sure whether it is finite), so it is difficult to decide what value to use even for a given m-function. In spite of this difficulty there is one method that seems to be intuitively highly satisfactory.

Let us first consider the case that the number of things in the world is infinite. This case has been usually handled (e.g., by Carnap) by calculating the m-values for all finite n and taking the limit as n tends to infinity. These results, however, cannot be used, because any two universal hypotheses have the same m (namely 0) and so does any contradictory hypothesis. These theories always give counter-intuitive results for the infinite case. We must face the fact that only the m-functions for finite n are useable. The method suggested by this consideration is a method of asymptotic comparison. We say that $m(A_1)$ is greater than $m(A_2)$ if for all sufficiently high n the value of the former is greater than the one of the latter. We will discuss this method in detail later, and we will show that it is a refinement of the use of limits.

Now let us suppose that the actual number of things in the world is N, a finite number. Then the correct m-value would be obtained by letting n equal N. But we do not know what N is. Yet we know that it is a very large number, much larger than any number of things used in an experiment. The best we

can do, due to our ignorance, is to require that our statements should hold for all sufficiently high n, and hope in any particular case that N is sufficiently high. (And if at some future time we get the exact or approximate value of N, we can always add a postulate to the effect that there are exactly N things in the world, and then our asymptotic comparison method will give the same results as if we had set n equal to N.)

How are we to build up F from a given m-function? F is to depend on H and E and their logical relation to each other. As we have said before, the content of the w.f.f. and their relationship can be expressed in terms of the m-values of combinations of H and E. Fortunately, we need not consider all combinations; since the m-value of the disjunction of two incompatible w.f.f. is the sum of the m-values, it will suffice to consider the strongest conjunctions of H and E because all other combinations are expressible as disjunctions of these. So we get four basic m-values in terms of which all relevant relations are expressible:

$$(1) \qquad x = m(H \cdot E) \qquad y = m(H \cdot \bar{E}) \qquad z = m(\bar{H} \cdot E) \qquad t = m(\bar{H} \cdot \bar{E})$$

The reader can easily check other combinations: $m(H) = x + y$, $m(E) = x + z$, $m(H \vee E) = x + y + z$, etc. And even these four values are not independent, since the disjunction of the w.f.f. is analytic,

$$(2) \qquad x + y + z + t = 1.$$

Since F is to depend exclusively on H and E and their logical relation to each other, F is a function that can be represented as depending on the quantities in (1) only.

CA6. $F(H, E)$ is a function $f(x, y, z, t)$ where x, y, z, t depend on H and E (according to (1)) and on n (since m depends on it).

We wish to state the method of comparison (see above) as it applies to F. Let us suppose that we have constructed an F-function on the basis of the selected m-function. Let us denote the value for a particular n by $F(H, E, n)$. We introduce the ordering principle:

(3) $F(H, E)$ is greater than $F(H', E')$ if there is an N such that for all n greater than N, $F(H, E, n)$ is greater than $F(H', E', n)$.

Equality and the relation of being less than are defined similarly. This definition seems to be the most rigorous and intuitive definition that we can build on our m-function. Some people may, however, find it distasteful because the values of F are functions of n, not numbers. It also has the disadvantage that if the curves of two functions intersect an infinite number of times, then the F-values are *incomparable*. This case seems to occur only in certain unusual types of statements not used in the sciences, but it is a disadvantage. Both of these difficulties can be overcome—at a price—by passing to the limit in F.

(4) Let $F^*(H, E)$ be the limit of $F(H, E, n)$ as n tends to infinity, and use $F^*(H, E)$ in place of $F(H, E)$.

Passing to the limit in F is not quite as bad as doing the same in m, but one still gets into trouble with generalized sentences. (The reader interested in this problem should compare Carnap's corresponding results (sections 56ff., loc.

cit.).) So we have the choice between the two methods; the following obvious theorem establishes the relation between them.

THEOREM 1. If F_1 is greater than F_2 according to method (4), it is also greater according to (3).

We will show later by examples that the converse is not true: There are cases where (4) gives the same value for both F's, while (3) can distinguish between them. Hence (3) is a refinement of (4). Perhaps the best procedure is to use (4) with numerical values whenever it gives a sufficiently fine ordering, and use (3) whenever (4) gives too many equal values.

Let us sum up our formal considerations: The problem has been analyzed into two questions, the finding of a suitable measure function m, and the selection of the function f of four variables. If we have the answer to these two questions, we have our F-concept. Given an H and an E, using m we can calculate the values x, y, z, t according to (1) for any fixed n, and then $f(x, y, z, t)$ is $F(H, E, n)$. Then we proceed according to (3) or (4) in the ordering of F's.

2. Material Conditions of Adequacy. Our task now is to set down all our intuition tells us about F, to be able to select an m and an f. The first six CA were really conventions; the material conditions are yet to follow. Our aim is completeness and no special attempt will be made to make conditions independent of each other. We will omit an intuitive condition only if it is immediately clear that it follows from previous conditions of adequacy.

First of all we must clarify the nature of H and E. We said that any w.f.f. is admitted. Actually we are interested in only those w.f.f. which express propositions that could possibly be of scientific interest. So in our formal development if we find exceptions to our theorems which are of no scientific interest we will simply ignore them. This is a weakening of CA1, but we do not try to state it precisely because of the difficulty of stating exactly what w.f.f. are of scientific interest. But to be safe we will ignore w.f.f. only if they are clearly uninteresting, such as analytic and contradictory w.f.f. At this time we must consider another weakening of CA1. We cannot handle a *statement of infinity*, i.e., a w.f.f. which cannot be satisfied by a finite number of individuals because it is not true in any finite s.d., so its m-function is identically 0. This means that there is no way to distinguish between it and a contradictory w.f.f. This is just one more shortcoming added to the fact that we have an oversimplified language. So we now reinterpret CA1 to apply only to such w.f.f. as are of scientific interest, and such that no combination of H and E is a statement of infinity. The remaining w.f.f. still cover a lot of ground; thus, e.g., since we allow dependencies, we have a richer language than that used by the authors writing about C.

Let us also examine more closely the meaning of our weights (p. 310). Since the number of s.d. varies with n, the weights will certainly depend on n also. Could the weights depend on E?. This would mean that they are not a priori determined, but empirically. This may or may not be desirable (for example Helmer and Oppenheim require it, while Carnap rejects it); at any rate we allow it in order to give ourselves a maximum amount of freedom. But we see no possible interpretation under which the weights would depend on H.

CA7. The weights depend on n, they may depend on E, but they must be independent of H.

We now turn to some intuitive conditions about greater and lesser factual support. First of all we give material content to the part of CA2 dealing with "best possible support." The evidence definitely proves that the hypothesis is true just in case the hypothesis is a logical consequence of the evidence.

CA8. $F(H, E)$ is 1 if and only if $\vdash E \supset H$.

The evidence definitely disproves the hypothesis just if it logically implies its negation, so this corresponds to $F = -1$. But we need no new condition for this since it follows from CA8 by means of CA3.

These were extreme cases. We now turn to intermediate values of F. Let us start with H_1 and E. If we now strengthen H_1 into H_2, without any additional supporting evidence, then F_2 should be less than F_1. First of all we have that $\vdash H_2 \supset H_1$, but not conversely (meaning of 'stronger'). 'Without additional supporting evidence' could mean that E is the same in both cases, but this is not enough. It could still be that E contains some information irrelevant for H_1 but supporting H_2. So we must assure that what E and H_1 have in common is at least as much as what E and H_2 have in common. (The information two w.f.f. have in common is expressed by their disjunction.)

CA9. If $\vdash H_2 \supset H_1$ and $\vdash [H_1 \vee E] \supset [H_2 \vee E]$, but not $\vdash H_1 \supset H_2$, then $F(H_1, E)$ is greater than $F(H_2, E)$.

Let us now consider the converse case where we strengthen the hypothesis, but the new part has very good support. More precisely, let us strengthen H_1 (whose F is neither -1 nor 1) into H_2 in such a manner that the new part of H_2 (the part in H_2 and not in H_1 : $H_2 \vee \bar{H}_1$) gets the best possible support from E; then F must increase.

CA10.[7] If $\vdash H_2 \supset H_1$ and $\vdash E \supset [H_2 \vee \bar{H}_1]$, but not $\vdash H_1 \supset H_2$ and not $\vdash E \supset H_1$ and not $\vdash E \supset \bar{H}_1$, then $F(H_2, E)$ is greater than $F(H_1, E)$.

Let us now turn to the middle point, where $F = 0$. According to CA2 this happens when the evidence is equally favorable and unfavorable (which very often means an irrelevant evidence). This concept is somewhat like that of the independence of two events in probability theory. Let us try to formulate an "independence condition" for H and E in this sense.

$$p(H \cdot E) = p(H) \times p(E). \tag{5}$$

[7] This CA serves to illustrate the relation between F and C. We see that F does not satisfy one of the fundamental C-principles, namely that if H_1 follows from H_2, then its C is at least as great as that of H_2 with respect to any evidence. Consider the historical example: H_1 is the bending of light-rays in a gravitational field, while H_2 is the General Theory of Relativity, and the evidence is the evidence *before* 1919. Certainly $F(H_2, E)$ is much greater than $F(H_1, E)$ since H_1 was not supported by any observation up to that time. Yet if we assigned say 80% credibility to H_2, we would have to assign at least that much to any of its consequences. Here is a case where H_1 has a high credibility (C) in spite of a very low degree of factual support (F)! The reasoning must be as follows: On the basis of its F-value and other considerations, simplicity of the hypothesis being the most important in this case, we assign by induction a high C-value to H_2, and consequently H_1 must get a C-value at least as high.

But what is to take the place of p in our case? Should we choose m? But that cannot be true, because then different choices of m lead to different formulations of our condition, while the 0-values of F are intuitively fixed. So we must find a unique p-function which when inserted in (5) will give an intuitive definition of the case when the evidence is equally favorable and unfavorable.

We will derive this p-function for languages without dependencies. Consider two independent atomic sentences A and B. Two atomic statements which are logically independent cannot support each other factually since they express distinct facts; so B as evidence is irrelevant to A, and $p(A \cdot B)$ is the product of $p(A)$ and $p(B)$. Similarly we get that the p-value of the conjunction of any number of independent atoms is the product of the individual p-values.

Now let us take another example. Consider an evidence E of the form $[A \cdot B] \vee [\bar{A} \cdot \bar{B}]$. This asserts that A is equivalent to B (where B is a logically independent atom). If this is all that we know, then it is just as favorable to A as to $\bar{A}$, hence it is irrelevant. So A and E must satisfy (5). But $A \cdot E$ is equivalent to $A \cdot B$, so we get that $p(A \cdot B) = p(A) \times p(E)$. From the above we know, however, that $p(A \cdot B) = p(A) \times p(B)$, hence $p(B) = p(E)$. When we apply (5) to $\bar{A}$ and E, we get that $p(\bar{B}) = p(E)$. So an atom and its negation have the same p, which must be $\frac{1}{2}$. Therefore the p-value of an s.d. is $(\frac{1}{2})^{\text{no. of atoms}}$ which is the same for all s.d. This describes p uniquely: It is a probability function sometimes called Wittgenstein's. We will use 'p' from now on only as a name for this particular probability function which assigns equal probability to all s.d. We can now use (5) for our purpose,

CA11. $\quad F(H, E) = 0$ if and only if $p(H \cdot E) = p(H) \times p(E)$.

Strictly speaking our argument applies only to languages without dependencies, and even then it makes use of an analogy. But we can show that there is a second approach for describing the 0-values, which applies to any language, and which leads to CA11. We prove this in the form of a theorem.

THEOREM 2. $F(H, E) = 0$ if and only if the s.d. in which H is true and the s.d. in which E is true have 0 correlation.

Proof. Take a fixed n. Enumerate all the s.d., and put a 1 under an s.d. if H is true in it, a 0 otherwise, under these put a 1 if E is true in the s.d., and 0 otherwise. We must now correlate the two rows of 1's and 0's. Let S be the total number of s.d., and h and e variables for the two rows of values. Then the correlation is

$$\frac{S\Sigma(he) - (\Sigma h \times \Sigma e)}{\sqrt{(S\Sigma h^2 - (\Sigma h)^2)(S\Sigma e^2 - (\Sigma e)^2)}}. \tag{6}$$

The sum of the h's gives us just the number of s.d. in which H is true, which can be simply written as $S \times p(H)$. (This follows from the definition of p.) Similarly the sum of the e's is $S \times p(E)$ and the sum of the he's is $S \times p(H \cdot E)$. Insert these values in (6) and simplify,

$$\frac{p(H \cdot E) - p(H) \times p(E)}{\sqrt{p(H) \times p(\bar{H}) \times p(E) \times p(\bar{E})}}. \tag{7}$$

From this we see that the correlation is 0 just when (5) holds, hence the theorem follows from CA11.

This theorem gives strong support to our claim that we have correctly reproduced our intuition in CA11, and since it applies also for languages with dependencies, we feel justified in stating CA11 as a general condition of adequacy.

Now we have considered the three basic F-values, and we have considered cases where F increases and where it decreases; the only remaining case is the one in which the value of F remains unchanged. Here we start with a fixed H, and an E_1 which will be strengthened by the addition of E_2 in such a way that F is unchanged. This happens when the new evidence is irrelevant; but irrelevant to what? We might guess that it must be irrelevant to H and E_1, but we will show that this is insufficient. Take the w.f.f. $[A \cdot B] \vee [\bar{A} \cdot \bar{B}]$, used above, as E_1, and A as H. If we strengthen the evidence by adding B as E_2 (B is irrelevant to both H and E_1), then F changes from 0 to 1 since H follows from $E_1 \cdot E_2$. The reason for this is that E_2 is not irrelevant to the conjunction of H and E_1. We see that we have to require that the new evidence be irrelevant to all combinations of the hypothesis and the old evidence. This is analogous to the condition of complete independence and can be assured by any three independent conditins. We will require that the new evidence be irrelevant to H, to the old evidence E_1, and to their conjunction.

CA12. If

$$p(H \cdot E_2) = p(H) \times p(E_2), \text{ and}$$
$$p(E_1 \cdot E_2) = p(E_1) \times p(E_2), \text{ and}$$
$$p(H \cdot E_1 \cdot E_2) = p(H \cdot E_1) \times p(E_2),$$

then $f(H, E_1 \cdot E_2)$ is the same as $F(H, E_1)$.

This finishes our material conditions of adequacy. Let us now see whether we have put down everything we want to require of our explicatum. Carnap has an excellent chapter (Ch. 1, loc. cit.) on the problem of explication. On page 7 he enumerates the conditions a good explicatum must satisfy: It must be similar to the explicandum (which is assured by the material CA), it must be exact (assured by the formal CA), it must be fruitful (which we will show by proving theorems and working out examples), and it must be as simple as the other conditions permit. This last point we must still assure:

CA13. The definition of F must be as simple as the other 12 CA permit.

This condition differs from all the other CA in that it is not a condition specifically for F, but for any explicatum; we list it here for the sake of completeness. While the concept of simplicity has never been precisely defined, its application is perfectly clear in many simple examples. We will make use of CA13 only when its application is non-controversial.

3. The Definition of F. In the first part we showed that our problem consists in choosing the auxiliary functions m and f. In the second part we completed the list of conditions of adquacy that F must satisfy. It is now our task to use these CA to narrow down step by step the choice of m and f. We will do this by a series of simple theorems; the reader not interested in details is advised to read the theorems and skip the proofs.

A few preliminary remarks will simplify our work. First of all we will see in most of our theorems that the conditions determine the values of m and f only for certain values of the variables (those which can be obtained in actual examples). But these values are a dense set, which means that the only way m and f can be continuous is that we assume that the theorems hold for all values. Then, by CA13, we will require that this be so. Again, in some cases (e.g., theorem 11) the proof holds for practically all values, and again by simplicity we will require that the theorem should hold also for those values about which we can say nothing otherwise. Such techniques are quite general in Mathematics, and in most philosophical writings they are used without explicit mention. We state this once and for all; in the proofs we will be concerned only with establishing the theorem about sufficiently many values of the variables.

Secondly we note that whether we use (3) or (4), changing the first few values of $F(H, E, n)$ does not change the result. So we will be perfectly satisfied if our theorems hold for all sufficiently high n, and we don't even care if F is not properly defined for a few values of n. Let us apply this to H and E. Since they (and their negations) are not contradictory and not statements of infinity, they will be true in some (but not all) s.d. It is a property of the s.d. that if a w.f.f. is true in an s.d. for given n, then there is a corresponding s.d. for all higher n in which it is true. E.g., if a w.f.f. is true in 2 s.d. for some n, it will be true in at least 2 s.d. in all higher n.[8] It is very easy to see that any w.f.f. of interest to science will be true in more than one s.d. for very high n, and since we don't care what happens for a finite number of n, we may as well assume that our w.f.f. are true in more than one s.d. for each n. These considerations will simplify our proofs.

THEOREM 3. All our weights must be positive (i.e., non-zero).

Proof. Suppose some s.d. received 0 weight. Call this s.d. Z.

Case 1. E is true in Z. Take $E \cdot \bar{Z}$ as H, and consider $F(H, E)$. Any combination of H and E differs from the corresponding combination of E and E only in being true (or false) in Z. But Z has weight 0, hence all the m-values are the same as for $F(E, E)$. By CA8, $F(E, E) = 1$. So $F(H, E)$ must be 1, which contradicts CA8, since H is stronger than E.

Case 2. E is false in Z. Let Z_1 and Z_2 be two s.d. in which E is true, let H_1 be $Z \vee Z_1$ and H_2 be Z_1. Then all conditions of CA10 are satisfied. (The only one that is not immediately clear is the second one, but $H_2 \vee \bar{H}_1$ is equivalent to $\bar{Z}$, which follows from E). Hence $F(H_2, E)$ must be greater than $F(H_1, E)$. But we see again that the combinations can differ only as to being true in Z, and hence have the same m-value. This would mean that the two F's are equal, another contradiction. Hence Z cannot have 0 weight.

We have said that H and E cannot be analytic or contradictory, but their combinations could be. In the following we let B stand for a combination of H and E, and investigate the consequences of B being either analytic or contradictory.

THEOREM 4. If $\vdash B$, then $m(B)$ is 1 for all n; if $\vdash \bar{B}$, then $m(B)$ is 0 for all n; otherwise the values of $m(B)$ lie between 0 and 1.

Proof. If $\vdash B$, then B is true in all s.d., hence $m(B)$ is 1 for all n, by CA5. If $\vdash \bar{B}$, then B is false in all s.d., hence its m must always be 0. If neither is the case, then B is true in some "model," and false in some other "model."[9] And if neither B nor $\bar{B}$ is a statement of infinity—which we require—then each is true in some finite "model," which can be represented by a s.d. And the same is true for all higher n. Since all weights are positive, this means that the values of $m(B)$ lie between 0 and 1 for all sufficiently high n, which is all we care about.

[8] Church, Alonzo, *Introduction to Mathematical Logic—Part I*, Princeton: Princeton University Press, 1944, p. 91.

[9] A theorem due to Gödel, ibid., pp. 73ff.

THEOREM 5. $f(x, y, z, t)$ is a continuous function defined for all values of the variables for which $x + y + z + t = 1$, and none of the following is 0: $x + y$, $x + z$, $z + t$, $y + t$.

Proof. This theorem is mainly a summing up of previous considerations. The fact that none of the four sums can be 0 is seen as follows: $x + y = m(H)$, so by theorem 4 it is 0 only when H is contradictory, which is excluded. Similarly the other three 0-values correspond to the forbidden cases of contradictory E, and analytic H and E respectively.

THEOREM 6. The values of f lie between -1 and 1.

THEOREM 7. $f(z, t, x, y) = -f(x, y, z, t)$.

THEOREM 8. $f(x, y, z, t) = 1$ if and only if $z = 0$.

Proof. These three theorems follow from CA2, CA3, and CA8 respectively. The first two are immediate. The last one is seen when we remember that $\vdash E \supset H$ just when $\vdash [\overline{\bar{H} \cdot E}]$, then it follows from theorem 4.

THEOREM 9. If x is increased and z decreased by the same amount, then $f(x, y, z, t)$ increases.[10]

Proof. We will show that this is the mathematical content of CA9. Going from $F(H_2, E)$ to $F(H_1, E)$ in this CA, E remains the same, so that $x + z$ is unchanged. The first two conditions of the CA assure that $H_2 \vee E$ is equivalent to $H_1 \vee E$, hence $x + y + z$ is unchanged. Therefore, t is unchanged, since the sum of the variables is 1. And if $x + z$ and $x + y + z$ are both unchanged, so is y. The first and third conditions assure that H_1 is true in more s.d. than H_2, hence $x + y$ is increased, which means that x is increased. And since the sum must remain 1, z must have decreased by the same amount. And CA9 tells us that this change results in an increase of F.

THEOREM 10. If t increases and y decreases by the same amount, then $f(x, y, z, t)$ increases; at least when x and z are both positive.

Proof. This is quite analogous to the previous proof, using CA10 in place of CA9. Passing from H_1 to H_2 in CA10 corresponds to an increase in t with an equal decrease in y; while the last two conditions in the CA mean that z and x are both positive. So the CA tells us that f must increase.

THEOREM 11. Any two s.d. in which E is true get the same weight, and any two s.d. in which E is false get the same weight.

Proof. This will be deduced from CA11. Select an H_1 so that $F(H_1, E)$ is 0. Let Z_1 be an s.d. in which E and H_1 are both true, while Z_2 and s.d. in which E is true but H_1 is false. Then $F(H_2, E)$ will also be 0 if we take $[H_1 \cdot \bar{Z}_1] \vee Z_2$ as H_2. (This follows from the fact that H_2 differs from H_1 only in that it is true in Z_2 but false in Z_1, hence all the p-values are the same.) Now let us suppose that the two s.d. got different weights, say Z_2 has a greater weight than Z_1. That means that $m(H_2 \cdot E)$ is greater than $m(H_1 \cdot E)$, while $m(\bar{H}_2 \cdot E)$ is less than $m(\bar{H}_1 \cdot E)$ by the same amount. Then by theorem 9 the two F's cannot be the same, which is a contradiction. The proof for the s.d. in which E is false is quite analogous. Constructing H_1 and H_2 as above, from Z's in which E is false, we get again that on the one hand the F's should be 0, on the other hand by theorem 10 they must be different; leading again to a contradiction if the s.d. get different weights.

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[10] Theorems 9 and 10 hold, of course, only if the variables remain within the bounds specified in theorem 5.

This theorem is precisely the result we needed to be able to translate CA11 into a condition on f. But first we must put the theorem into a more useful form. Let us call the weights of the s.d. in which E is true, w. The other weights (those of s.d. in which E is false) are also all equal and positive, so we can call them qw, q being the quotient of the two different weights. Using the fact that the number of s.d. in which A is true is $Sp(A)$, we see that the sum of the weights is $wSp(E) + qwSp(\bar{E})$. Since this sum is 1, we get a formula for w,

$$w = 1/S[p(E) + qp(\bar{E})]. \tag{8}$$

So if we know q, we can get w, and hence all the weights. q may, of course, depend on E and on n; but we have reduced the problem of the m-function to the finding of q. Let us denote the quantity $p(E) + qp(\bar{E})$ by D, then $w = 1/SD$. Given an arbitrary w.f.f. A, we have to break up the s.d. in which it is true into those that get weight w and those that get weight qw. There are $Sp(A \cdot E)$ of the former, $Sp(A \cdot \bar{E})$ of the latter. So we get, using (8) and then (1),

$$m(A) = [p(A \cdot E) + qp(A \cdot \bar{E})]/D. \tag{9}$$

$$\begin{aligned} x &= p(H \cdot E)/D, & y &= qp(H \cdot \bar{E})/D, \\ z &= p(\bar{H} \cdot E)/D, & t &= qp(\bar{H} \cdot \bar{E})/D. \end{aligned} \tag{10}$$

Using these values we find that the equation (5) can be translated into $x = (x + y)(x + z)$. (This can be checked by a routine calculation.) And then we can translate CA11 into:

THEOREM 12. $f(x, y, z, t) = 0$ if and only if $x = (x + y)(x + z)$.

THEOREM 13. f is the ratio of two functions both homogeneous in x, y, z, t to the degree k;[11] where k is the smallest integer permissible under the first 12 CA.

Proof. Since f is a normalized function, it is simplest to take it as the ratio of two similar functions. Since x, y, z, t play analogous roles in all definitions, the functions must be homogeneous in the variables. And since the normalization of m is purely conventional, f must be unchanged if all variables are multiplied by the same number, hence the numerator and the denominator must be homogeneous to the same degree. Finally, smaller k-values give simpler functions. So this theorem is our interpretation of what CA13 tells us about f.

THEOREM 14. q, the ratio of weights of the s.d. in which E is false to those in which E is true, is independent of E and n. Hence each permissible m-function is determined by the choice of a positive constant q.

THEOREM 15. $f(xP, y + x(1 - P)q, zP, t + z(1 - P)q) = f(x, y, z, t)$ where P is any number between 0 and 1, and q is the number that characterizes the m-function.

Proof. Both theorems follow from CA12. Let us examine this CA precisely. It tells us that under certain kinds of changes the value of f is unchanged. We can translate the conditions of change into conditions on x, y, z, t. Let us denote the values of the variables after the change by x', y', z', t'. We can use (10) to calculate both sets of variables, they differ only in that E is E_1 in the first case, $E_1 \cdot E_2$ in the second case. Making use of the

[11] A function is homogeneous to degree k if multiplying its arguments by c multiplies the value by c^k.

conditions in CA12, we can express the new variables in terms of the old ones; this is purely routine, and we state only the result. Let R denote the ratio of denominators, and P the value of $p(E_2)$, then

$$x' = xPR \qquad y' = yRq'/q + x(1 - P)Rq' \qquad z' = zPR \tag{11}$$
$$t' = tRq'/q + z(1 - Q)Rq'.$$

First of all we note that all variables have been multiplied by R, but by theorem 13 this leaves f unchanged. Cross them out. Then (11) expresses a transformation that must leave f unchanged. How is this possible? The terms in y' will be like those in y if we factor out q'/q, and similarly for t' and t. But that leaves the x' and z' terms. Somehow terms in xP must combine with terms in $x(1 - P)q'$, z being analogous. But this is impossible in general, since P depends on E_2, while q' in general depends on $E_1 \cdot E_2$. The only possible way out is that q' is independent of the evidence. That makes it possible to choose f in such a way that these terms combine in the right manner. But since the form of f is independent of n, q must be independent of n also to allow the combination. This proves theorem 14. (11) is now considerably simplified: R dropped out, and $q = q'$ is a constant. Theorem 15 states exactly that f is unchanged by this transformation, hence it follows from CA12.

It is interesting to note that the last theorem is the first we have which establishes different conditions on f for different choices of the m-function!

The reader can check that we have now translated our CA into conditions on m and f, in accordance with our program. Our m-functions have been determined to be the members of a small class (see theorems 3, 11, 14), each m determined by one positive constant q. Let us now try to determine the possible f's. According to theorem 13 we must find the smallest integer k such that the ratio of two homogeneous functions of degree k will satisfy the CA (which are translated in theorems 5–10, 12, 15). And since one of the conditions of f (theorem 15) depends on q, the form of the f-function may depend on the choice of m. We will now systematically consider larger and larger values of k, till we find a definition satisfying all our conditions.

THEOREM 16. k cannot be 1.

Proof. If k is 1, the numerator of f is of the form $ax + by + cz + dt$ (and the denominator is similar). This can be eliminated by a single condition. Consider theorem 12. Let us choose as sets of values for the variables: (1/2, 1/4, 1/6, 1/12); (1/2, 1/3, 1/10, 1/15); (1/2, 3/8, 1/14, 3/56); and (1/4, 1/4, 1/4, 1/4,). All of these satisfy the condition of theorem 12, and hence f must be 0. So the numerator of f must vanish for these four sets of values. But simple algebra shows that this is only possible if a, b, c, d are all 0, which is forbidden.

THEOREM 17. For each m-function there is a unique f-function with $k = 2$.

Proof. If $k = 2$, f has the form

$$\frac{ax^2 + bxy + cy^2 + dxz + eyz + gz^2 + ixt + jyt + rzt + st^2}{\bar{a}x^2 + \bar{b}xy + \bar{c}y^2 + \bar{d}xz + \bar{e}yz + \bar{g}z^2 + \bar{i}xt + \bar{j}yt + \bar{r}zt + \bar{s}t^2}. \tag{12}$$

We must now try to satisfy all our f-theorems by a suitable choice of the 20 coefficients. Take theorem 7. It tells us that under a certain interchange of variables f changes its sign. This must mean that the numerator changes sign and the denominator is unchanged (since the denominator cannot be 0). This means that d and j must be 0; g, s, r, e are the negatives of a, c, b, i; and $\bar{g}$, $\bar{s}$, $\bar{r}$, $\bar{e}$ are equal to $\bar{a}$, $\bar{c}$, $\bar{b}$, $\bar{i}$ respectively. This cuts the coefficients down to 10.

Take theorems 6 and 8. If $z = 0$, f is 1, otherwise f is less than 1. (The fact that f is at least -1 follows from this by theorem 7.) Write down the formula for f, setting z equal to 0, and equate it to 1. Since this is to hold identically, we get the coefficients in the numerator equal to the corresponding ones in the denominator, and c, $\bar{c}$, $\bar{j}$ equal to 0. (Furthermore, f must be less than 1 for positive z, but this will be clearly so in (14) for positive u, so we need to add nothing.) This gives

$$f(x, y, z, t) = \frac{a(x^2 - z^2) + b(xy - zt) + i(xt - yz)}{a(x^2 + z^2) + b(xy + zt) + i(xt + yz) + \bar{d}xz}. \tag{13}$$

Take theorem 12. Work out the examples mentioned in theorem 16. They show that a and b must be 0. On the other hand whenever the condition of theorem 12 holds, $xt - yz$ is 0, so it is both necessary and sufficient to set a and b equal to 0. This reduces (13) to a simpler form. (Since i cannot be 0 now, we divide by it, putting u for $\bar{d}/i$.)

$$f(x, y, z, t) = \frac{xt - yz}{xt + yz + uxz}. \tag{14}$$

Theorem 5 is true clearly just in case u is positive. A little more mathematical checking will convince us that theorems 9 and 10 do not impose any new conditions at all (probably due to the fact that they were used indirectly in determining the form of the m-functions). Take our last theorem, number 15. Perform the transformation called for in this theorem on (14) and equate the result to (14). This equality will hold in complete generality just in case u is set equal to $2q$. So for a given m-function (hence given q) we get the unique f-function satisfying all conditions:

$$f(x, y, z, t) = \frac{xt - yz}{xt + yz + 2qxz}. \tag{15}$$

Therefore, if we use theorem 13, we must choose (15) as a definition of f. This still leaves us one choice, the choice of q; q determines m by (10), and it also determines f by (15). Let us combine these two equations by substituting the values of the variables from (10) into (15). (We have now solved the two problems posed at the end of Part 1, and we are ready to define F.) When we make this substitution we find, to our surprise, that all quantities containing q cancel out. So it makes no difference which m-function we select, we get the same *unique* f-function. Let us sum this up:

THEOREM 18. There is a unique explicatum satisfying all the CA, namely

$$F(H, E) = \frac{p(H \cdot E)p(\bar{H} \cdot \bar{E}) - p(H \cdot \bar{E})p(\bar{H} \cdot E)}{p(H \cdot E)p(\bar{H} \cdot \bar{E}) + p(H \cdot \bar{E})p(\bar{H} \cdot E) + 2p(H \cdot E)p(\bar{H} \cdot E)}. \tag{16}$$

It may be worth-while stating an equivalent form of this. If we introduce the usual definition for a relative probability, namely $p(A, B) = p(A \cdot B)/p(B)$, we can write (16) as (we omit the routine proof):

$$F(H, E) = \frac{p(E, H) - p(E, \bar{H})}{p(E, H) + p(E, \bar{H})}. \tag{17}$$

This gives us a complete theory of the explication of *degree of factual support* for our simple language.

4. Examples.[12]

Example 1. We want to illustrate the fact that it makes no difference which

[12] In order to avoid quotation marks, we will use w.f.f. as names for themselves.

m-function we choose. Let H be $P(a) \cdot P(b)$, while E is $P(a)$. We choose the values 3, 1, ½ for q, and for each value we calculate the values of the variables by means of (10), and then F by (15). The reader is advised to write out the three f-formulas given by (15) for the three values of q. x, y, z, t equal (1/8, 0, 1/8, 3/4); (1/4, 0, 1/4, 1/2); (1/3, 0, 1/3, 1/3) respectively. In all three cases we get $F(H, E) = \frac{1}{2}$.

In the remaining examples we will just calculate values of F according to one formula, a convenient equivalent form of (16). This form has the advantage that it is determined by three values: $p(H)$, $p(E)$, $p(H \cdot E)$.

$$F(H, E) = \frac{p(H \cdot E)/p(H) - [p(E) - p(H \cdot E)]/[1 - p(H)]}{p(H \cdot E)/p(H) + [p(E) - p(H \cdot E)]/[1 - p(H)]}. \tag{18}$$

The examples were selected to illustrate important theoretical points. The details of calculation will be omitted. In calculations one should consider all s.d. of the language, but this complicates matters too much. Fortunately, if a set of atomic sentences is independent of all others in the language, the result of the calculation is the same if we ignore all other atoms. So in all examples it is assumed that the atoms occurring are independent of all other atoms in the language.

Example 2. $F(P(a), P(a) \vee P(b)) = 1/3$. Weaker support than in Ex. 1.

Example 3. $F(P(a), P(a) \cdot P(b)) = 1$. Evidence proves hypothesis is true.

Example 4. $F(P(a), \bar{P}(a) \cdot P(b)) = -1$. Evidence proves hypothesis is false.

Example 5. $F(P(a), P(b)) = F(P(a), Q(a)) = 0$. Irrelevant evidence.

Example 6. $F(P(a) \vee P(b), P(a)\bar{P}(b) \vee \bar{P}(a)P(c)) = 0$. The evidence is equally favorable and unfavorable.

Example 7. $F(P(a) \cdot P(b) \cdot P(c), P(a)) = .40$.
$F(P(a) \cdot P(b) \cdot P(c), P(a) \cdot P(b)) = .75$. Increasing the supporting evidence.

Example 8. $F(P(a) \cdot P(b) \cdot P(c) \cdot P(d), P(a) \cdot P(b)) = .67$. Compare with the previous example. We added a new unsupported hypothesis to a well-supported hypothesis, and F was reduced from .75 to .67.

Example 9. $F(P(a) \cdot P(b), P(a) \cdot Q(a)) = \frac{1}{2}$. We took example 1 and added some totally irrelevant evidence.

Example 10. This is the first example where we must take account of the dependence on n. (So far the f-functions had the same value for all n.) Let H be $(x)P(x)$, E be $P(a)$. $p(H) = p(H \cdot E) = (\frac{1}{2})^n$, $p(E) = \frac{1}{2}$. Substitute these values into (18); by a process of approximation we get for large n, $F(H, E) = 1/3 + 4/9(\frac{1}{2})^n + \cdots$ Or if we are interested only in the limit, $F^*(H, E) = 1/3$.

Example 11. $F^*((x)P(x), P(a_1) \cdot \cdots \cdot P(a_k)) = (2^k - 1/2^k + 1)$. So we see that although F^* cannot be 1 if our hypothesis is universal and the evidence is singular, it approaches 1 as k increases. We note also that the *rate* of increase decreases, so that each new piece of evidence adds less to the support than the previous piece.

Example 12. $F^*((\exists x)\bar{P}(x), P(a_1) \cdot \cdots \cdot P(a_k)) = -(2^k - 1/2^k + 1)$. This is

the converse of the previous example. The evidence is again singular, but the hypothesis is existential. While the evidence never disproves the hypothesis, F^* tends to -1 with increasing k.

Example 13. $F((x)P(x), \bar{P}(a)) = -1$. A single fact can disprove a universal hypothesis.

Example 14. $F((\exists x)\bar{P}(x), \bar{P}(a)) = 1$. A single fact can prove that an existential hypothesis is true.

Example 15. $F((x)(P(x)\cdot Q(x)), (\exists x)P(x)) = \frac{1}{2}(\frac{1}{2})^n + \frac{1}{4}(\frac{1}{2})^{2n} + \cdots$
$F((x)(P(x)\cdot Q(x)), (\exists x)P(x)\cdot(\exists x)Q(x)) = (\frac{1}{2})^n + \frac{1}{2}(\frac{1}{2})^{2n} + \cdots$
This is an example demonstrating that method (3) is really stronger than method (4), see p. 311, F^* is 0 in both cases, but according to (3) the second F is greater, since for high n (when we can neglect all but the first term) it has greater values.

Example 16. In this example we choose a two-place predicate. This is also the first example where we allow dependencies between our atomic sentences. Let H be $(x)(y)R(x, y)$, and let E be $R(a, b)$. Take first R as a general predicate: there are no dependencies. Then we get $F(H, E) = 1/3 + 4/9(\frac{1}{2})^{n^2} + \cdots$. If we now take for R a reflexive predicate (the postulate $(x)R(x, x)$ being added to the logical axioms, see p. 309), then $F(H, E) = 1/3 + 4/9\cdot 2^n\cdot(\frac{1}{2})^{n^2} + \cdots$. F^* is 1/3 in each case, but by (3) the second F is greater. We can interpret this by saying that in the second case there are atomic sentences, like $P(a, a)$, which need no support since they follow from our postulate; a distinction that is lost in (4).

Example 17. Take a case of mixed operators. Let H be $(x)(\exists y)R(x, y)$, E be $R(a, b)$. Then $F(H, E) = 1/2n + \cdots$, while $F^*(H, \bar{E}) = -1/2n + \cdots$. In both cases the support is very slight (F^* is 0), favorable in the former, unfavorable in the latter.

Example 18. Let us consider a very general example; the case where all the evidence is favorable. This means that $\vdash H \supset E$. Then $H\cdot E$ is equivalent to H. Using this fact in our formula (18), we can get the simple form $\dfrac{1 - p(E)}{1 + p(E) - 2p(H)}$. If we introduce the concept: *strength* $(A) = 1 - p(A)$, we can rewrite this in the form

$$\frac{\text{strength (E)}}{2 \text{ strength } (H) - \text{strength } (E)}.$$

For a given hypothesis stronger evidence gives better support; for a given evidence the stronger hypothesis gets weaker support. This is in excellent agreement with our intuition.

Example 19. We will not discuss the problem of how statistical statements are to be expressed in our simple language. We just assume, for the sake of the example, that this can be done. Let H_r express for each n that rn of the n things has the property P (r between 0 and 1),

while E_{sm} states about m things that sm of them are P's, while the others are not (s between 0 and 1, m less than n). Then

$$F^*(H_r, E_{sm}) = \frac{(2r^s(1-r)^{1-s})^m - 1}{(2r^s(1-r)^{1-s})^m + 1}.$$

For a given E_{sm} we find that the hypothesis given the best support is the one having $r = s$, and in general this support increases with the size of the sample, m. This is as it should be. So, for example, if we toss a coin 100 times and get 60 heads, this evidence supports the hypothesis that just 60% of all tosses will be heads best, and the support is $F^* = .77$.

Example 20. Finally we want to discuss the relation between F and the correlation coefficient (see p. 314). For this purpose we write F in the form: $[p(H \cdot E) - p(H)p(E)]/[p(H)p(\bar{H} \cdot E) + p(\bar{H})p(H \cdot E)]$. Compare this with (7). We see that they have the same numerator; hence F is positive, negative, or 0 just when the correlation coefficient is. But the latter is otherwise quite different from F; we see this already by noting that it is symmetric in H and E, while F, of course, is not. So, e.g., the correlation is 1 only if $F(H, E)$ and $F(E, H)$ are *both* 1.

5. Summary. We have completed our explication of *degree of factual support* by stating in thirteen clear conditions all that our intuition told us about the concept and then showing that there is a unique F-function satisfying these conditions. This approach to the problem of explication has the great advantage (assuming that we have made no purely mathematical mistake) that any objections must be directed towards our conditions of adequacy, and hence the dispute will be clearly defined. There are two possible types of objections which we will very briefly consider.

If someone rejected one of our CA, there would be no hope for agreement. But at least we would find the root of our disagreement; and it would be his task to formulate alternate CA, and find his F-function. Then anyone wishing to use F could choose between the two sets of CA. The most we can say is that the 13 CA here stated express *our* intuition.

Someone could accept all our conditions but argue that there are other necessary conditions which do not follow from ours. Then he can use all our theorems except the last one; in place of that he will find that no function with $k = 2$ can be used, and he will have to find the lowest k for which not only our but his CA are also satisfied. Then someone wishing to use F will have to decide whether the additional conditions are important enough to warrant the greater complexity of F.

It is important, however, that these differences should not be of a trivial nature, as for example objecting to one of our conventions. If someone decided, e.g., that F should range between 0 and 1, simply replace our F by $\frac{1}{2}(F + 1)$. Then 0 expresses disproof, $\frac{1}{2}$ irrelevance, and 1 still is the best possible support;

but nothing essential is changed. In place of (17) we then get $p(E, H)/p(E, H) + p(E, \bar{H})$.

One line of attack against many explicata came from people who insisted that the measure-functions used must be "empirical" (m must depend on E), or conversely from the advocates of a single, fixed m-function. It is very interesting, therefore, to note that one of our m-functions is independent of E, while all others depend on E. (If q is 1, all weights are equal, hence m is p. Otherwise the weights depend on which s.d. are the ones in which E is true.) So for our purposes this whole dispute is pointless since both types of measures lead to the same explicatum.

The language-system for which we formulated F in this paper is very primitive, and the most immediate problem for future research will be the extension of our definition to higher languages. There is really only one problem in this: If we can extend the concept of *state-description* to higher languages, then p is again the measure assigning equal weights to all these s.d., and (16) will define F. Some results for higher languages will be established in a forthcoming paper by Kemeny, replacing the concept of an s.d. by the more general concept of a finite model of the language.

Finally, it is hoped that the definition of F will be an aid in the explication of C. As we have said in the introduction, F is one of many factors influencing C, and an explication of the latter should start with definitions for all factors, and then (probably) proceed by the method of explication set forth in this paper. In this sense we hope that our paper will prove suggestive, in method as well as in content, for future work in this field.

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