Axiomatization of Qualitative Belief Structure

S. K. M. Wong, Y. Y. Yao, P. Bollmann, and H. C. Bürger

***Abstract*—There are important theoretical and practical reasons to study belief structures. Similar to qualitative probability, qualitative belief can also be described in terms of a preference relation. One of the objectives in this study is to specify the precise conditions that a preference relation must satisfy such that it can be faithfully represented by a belief function. Two special classes of preference relations identified are *weak* and *strict* belief relations. It is shown that only strict belief relations are consistent (compatible) with belief functions. More importantly, the axiomatization of qualitative belief provides a foundation to develop a utility theory for decision making based on belief functions. The established relationship between qualitative probability and qualitative belief may also lead to a better understanding and useful applications of belief structures in approximate reasoning.**

I. Introduction

THERE are two basic approaches (quantitative and qualitative) for representing uncertain information, and reasoning with such information [1], [3], [13], [16], [24]. In the quantitative approach, a number is associated with each proposition, namely, we express our *belief* in a proposition by a numeric value. On the other hand, in the qualitative approach, we merely state our *preference* on a set of propositions without exact quantification. Both these approaches are very useful for management of uncertainty. In fact, probability theory has been extensively studied within both the quantitative and qualitative frameworks [7], [8], [17].

One of the well known quantitative approaches, the theory of belief functions [18]–[21], has generated considerable interest in recent years. In this theory, belief may be interpreted as a generalization of probability. Belief functions are particularly useful in situations where input required by the Bayesian theory is not available. Although belief functions provide a useful and effective tool for the quantification of subjective, personal judgments, so far very little attention has been focused on the study of the structure of qualitative belief. There are many important reasons for a more detailed study of belief structure [6], [20]. It is the view of many researchers that humans frequently reason in qualitative rather than quantitative terms [2]. If the theory of belief is considered as a generalization of probability, one should not overlook the study of qualitative belief because the understanding of qualitative probability has played a very important role in the development of probability theory [7], [8], [10], [11], [17]. We believe that further investigations of belief structures will lead to deeper insights into the theory of belief functions as well. Another reason for studying qualitative belief is that it is much easier for humans to provide consistent qualitative rather than quantitative judgments [16]. Therefore, there are both practical and theoretical reasons to consider qualitative belief in approximate reasoning.

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Recently, Wong *et al.* [25] investigated the formal, axiomatic structure of qualitative belief. Qualitative judgments can be described by a preference relation that defines the relationship between different propositions. However, not every preference relation is necessarily consistent with quantitative belief. Specifically, they showed that if a preference relation satisfies a certain set of axioms, there exists a belief function that *partially agrees* with such a relation. In this study, we extend these axioms and show that if a preference relation satisfies the new set of axioms, there exists a belief function that *fully agrees* with the given relation. Based on these criteria, we identify two classes of preference relations: the *weak* belief relation and the *strict* belief relation. Another objective of this paper is to study the axiomatic characterization of belief relations. To expose the connection between qualitative probability and qualitative belief, we discuss the two views of interpreting belief relations, the allocation and compatibility views [12], [19], [21], [23]. In particular, it is explicitly shown that in the compatibility view, qualitative belief may be considered as a result of transferring qualitative probability from the evidence frame to the frame of interest [27].

II. Belief Functions And Belief Relations

For completeness, we briefly review the basic concepts of belief functions and belief relations. The former deals with the representation of quantitative belief, while the latter focuses on the representation of qualitative belief.

Suppose $\Theta = \{\theta_1, \ldots, \theta_n\}$is a finite set of all possible answers to a given *question* according to one's knowledge, and only one of these answers is correct. This set Θ is referred to as the *frame of discernment* or simply the *frame* defined by the question. Any subset $A \subseteq \Theta$ is regarded as a proposition, and the correct answer to the question lies in some proposition A. The power set 2^Θ of Θ denotes the set of all propositions discerned by the frame Θ. In a situation with incomplete information, it is not possible to say with certainty which proposition contains the correct answer. However, based on the evidence at hand it may be possible to express one's belief in different propositions. The theory of belief functions [18] provides a useful and effective tool for the description of the subjective, personal judgments in such situations.

A belief function Bel is a mapping from 2^Θ to the interval $[0, 1]$, Bel: $2^\Theta \to [0, 1]$, which satisfies the following axioms:

S1) $\mathrm{Bel}(\emptyset) = 0$,

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S2) $\mathrm{Bel}(\Theta) = 1$,

S3) For every positive integer n and every collection $A_1, A_2, \ldots, A_n$, of subsets of Θ, $\mathrm{Bel}(A_1 \cup A_2 \ldots \cup A_n) \geq \sum_i \mathrm{Bel}(A_i) - \sum_{i<j} \mathrm{Bel}(A_i \cap A_j) \pm \ldots + (-1)^{n+1}\mathrm{Bel}(A_1 \cap A_2 \ldots \cap A_n)$.

Axioms S1) and S2) state that the proposition $\emptyset$ is believed to be false (impossibility) and the proposition Θ is believed to be true (certainty). These two axioms are in fact the first two Kolmogorov axioms for defining probability [7]. Axiom S3) indicates that beliefs are superadditive, which become additive in the special case. Thus, belief functions can be viewed as a generalization of additive probability functions.

A belief function can be equivalently defined by another mapping, $m : 2^\Theta \rightarrow [0,1]$, which is called a basic probability assignment satisfying the axioms:

M1) $m(\emptyset) = 0$,

M2) $\sum_{A\in 2^\Theta} m(A) = 1$.

In terms of the basic probability assignment, the belief in a proposition $A \in 2^\Theta$ can be expressed as

$$\text{M3)} \quad \mathrm{Bel}(A) = \sum_{A \supseteq B} m(B),$$

where the summation is restricted to the elements B in 2^Θ, which are subsets of A. Conversely, by the Möbius inversion one can construct the basic probability assignment from a belief function [18]. Therefore, a belief function can be defined by the axioms S1)–S3) or M1)–M3). It should be noted that the *closed* world assumption has been implicitly used here in defining the belief functions [20]. Nevertheless, our results can be easily extended to generalized belief functions with the *open* world assumption [20], [26].

For a given belief function, one can define another function called *plausibility* as

$$\mathrm{Pl}(A) = 1 - \mathrm{Bel}(A^c),$$

where $A^c = \Theta - A$ denotes the complement of A. The belief in a proposition A is interpreted as the belief one actually commits to A, whereas the plausibility of A is interpreted as the maximum possible belief one may commit to A. It can be easily verified that $\mathrm{Pl}(A) \geq \mathrm{Bel}(A)$. Thus, the pair of belief and plausibility, $[\mathrm{Bel}(A), \mathrm{Pl}(A)]$, represents the quantitative judgments on a proposition A based on an evidence.

Following the development of comparative or qualitative probability [7], [17], Wong *et al.* [25] reported some preliminary results on the study of qualitative belief. In the qualitative approach, it is assumed that one is able to express one's preference or indifference on any two propositions $A, B \in 2^\Theta$. It is, however, not necessary to express numerically how much one prefers proposition A to proposition B. Formally, the qualitative judgments can be described in terms of a preference relation $\succ$. By $A \succ B$, we mean that *A is preferred to B*. In the absence of strict preference, i.e., $\neg(A \succ B)$ and $\neg(B \succ A)$, we say that A and B are *indifferent*, written $A \sim B$. We also write $A \succeq B$, if $A \succ B$ or $A \sim B$. The concept of preference relations provides a convenient way to represent qualitative judgments. However, whether such a preference relation is consistent with a belief function depends very much on the preference structure of the relation itself.

Definition 1: Let Θ be a frame, Bel a belief function from 2^Θ to $[0,1]$, and $\succ$ a preference relation on 2^Θ. We say that Bel *partially agrees* with $\succ$ if for $A, B \in 2^\Theta$,

$$A \succ B \Longrightarrow \mathrm{Bel}(A) > \mathrm{Bel}(B),$$

and that Bel *fully agrees* with $\succ$ if

$$A \succ B \Longleftrightarrow \mathrm{Bel}(A) > \mathrm{Bel}(B).$$

Moreover, we say that $\succ$ is *consistent* or *compatible* with Bel if Bel fully agrees with $\succ$.

Definition 2: Let $\succ$ be a preference relation on 2^Θ. We call the relation $\succ$ a *weak belief relation* if there exists a belief function, $\mathrm{Bel}{:}2^\Theta \rightarrow [0,1]$, which partially agrees with $\succ$. Similarly, we call the relation $\succ$ a (*strict*) *belief relation* or a *qualitative belief relation* if there exists a belief function, $\mathrm{Bel}{:}2^\Theta \rightarrow [0,1]$, which fully agrees with $\succ$.

The previous definitions characterize two classes of preference relations. If a preference relation $\succ$ is a belief relation, then $A \succ B$, for instance, may be interpreted as *the belief in A is more than the belief in B*. Similar to the definition of quantitative plausibility, we can also define qualitative plausibility in terms of a belief relation.

Definition 3: Let $\succ$ be a (weak) belief relation on 2^Θ. A (weak) plausibility relation $\succ'$ can be defined as

$$A \succ' B \Longleftrightarrow B^c \succ A^c.$$

Clearly, such a definition is compatible with thedefinition of quantitative plausibility.

In the following section, we will investigate the formal, axiomatic structure of these two classes of preference relations. In other words, we look for the conditions that characterize belief relations.

III. Axiomatization Of Belief Relations

We have identified two classes of preference relations, namely, the weak and the strict belief relations. One of the major concerns in the development of a belief theory is to establish the relationship between quantitative and qualitative representations of belief. It is therefore important to establish the precise axioms that a preference relation must satisfy in order to ensure the existence of a belief function that partially or fully agrees with such a relation.

Wong *et al.* [25] studied a class of preference relations characterized by the following axioms: for $A, B, C \in 2^\Theta$,

B1) (asymmetry): $A \succ B \Longrightarrow \neg(B \succ A)$,

B2) (negative transitivity): $(\neg(A \succ B), \neg(B \succ C)) \Longrightarrow \neg(A \succ C)$,

B3) (dominance): $A \supseteq B \Longrightarrow \neg(B \succ A)$..

Axioms B1) and B2) say that the preference relation $\succ$ is a weak order [8]. Axiom B3) specifies the relationship between two propositions when one is a subset of the other. From a rational point of view, if one commits more belief in A than in B, then one should not at the same time commit more belief in B than in A. If one does not commit more belief in A than in B, nor commits more belief in B than in C, then one should not commit more belief in A than in C. One should also not commit more belief in a subset than in the set itself. Thus, the previous axioms seem plausible for the characterization of a rational belief structure. However, axioms B1)–B3) are not sufficient to guarantee the existence of a belief function that fully agrees with the preference relation $\succ$. Nevertheless, these three axioms are sufficient to ensure the existence of a belief function that partially agrees with $\succ$, as indicated by the following theorem discussed in [25].

Theorem 1: Let $\succ$ be a preference relation on 2^{Θ}. If the relation $\succ$ satisfies axioms B1)–B3), then there exists a belief function Bel that *partially agrees* with $\succ$, namely, for $A, B \in 2^{\Theta}$,

$$A \succ B \Longrightarrow \mathrm{Bel}(A) > \mathrm{Bel}(B).$$

It should be noted that Theorem 1 provides a set of sufficient conditions on a preference relation $\succ$ for the existence of a belief function that partially agrees with $\succ$. The following example indicates that they are not necessary conditions.

Example 1: Let $\Theta = \{\theta_1, \theta_2, \theta_3\}$. Consider the preference relation:

$$\{\theta_1,\theta_2,\theta_3\} \succ \begin{matrix} \{\theta_1,\theta_2\} \\ \{\theta_2,\theta_3\} \succ \{\theta_1,\theta_3\} \end{matrix} \succ \begin{matrix} \{\theta_1\} \\ \{\theta_2\} \\ \{\theta_3\} \\ \emptyset. \end{matrix}$$

In the previous preference relation, we have adopted the convention that $A \succ B \succ C$ implies $A \succ C$. Obviously, this preference relation violates the axiom B2) of negative transitivity, because $\neg(\{\theta_2,\theta_3\} \succ \{\theta_1,\theta_2\})$ and $\neg(\{\theta_1,\theta_2\} \succ \{\theta_1,\theta_3\})$ but $\{\theta_2,\theta_3\} \succ \{\theta_1,\theta_3\}$. On the other hand, consider a belief function Bel:

$$\mathrm{Bel}(\{\theta_1,\theta_2\}) = 0.3, \mathrm{Bel}(\{\theta_1,\theta_3\}) = 0.3, \mathrm{Bel}(\{\theta_2,\theta_3\}) = 0.4$$

$$\mathrm{Bel}(\{\theta_1,\theta_2,\theta_3\}) = 1, \mathrm{Bel}(A) = 0 \text{ for all other } A \in 2^{\Theta}.$$

It can be easily verified that this belief function partially agrees with the preference relation given in this example, although it does not satisfy axiom B2).

For a special class of preference relations characterized by axioms B1) and B2), we have the following theorem [25].

Theorem 2: Let $\succ$ be a preference relation satisfying axioms B1) and B2). Then there exists a belief function Bel such that for $A, B \in 2^{\Theta}$:

$$A \succ B \Longrightarrow \mathrm{Bel}(A) > \mathrm{Bel}(B)$$

if and only if the dominance axiom B3) holds.

A question immediately arises: what axioms other than B1)–B3) should be satisfied by a preference relation $\succ$ such that there exists a belief function fully agreeing with it? In order to answer this question, we introduce two new axioms [4]: for $A, B, C \in 2^{\Theta}$.

B4) (partial monotonicity):$(A \supset B, A \cap C = \emptyset) \Longrightarrow (A \succ B \Longrightarrow A \cup C \succ B \cup C)$,

B5) (nontriviality): $\Theta \succ \emptyset$.

Axiom B4) is implied by the monotonicity axiom $(A \cap C = \emptyset, B \cap C = \emptyset) \Longrightarrow (A \succ B \Longleftrightarrow A \cup C \succ B \cup C)$ that has been used to characterize qualitative probability [8], [17]. Axiom B5) eliminates the trivial preference relation, namely, $A \sim B$ for all $A, B \in 2^{\Theta}$. We will first show that B1)–B5) form a set of independent axioms.

Theorem 3: B1)–B5) are independent axioms.

Proof: Let $\Theta = \{\theta_1, \theta_2\}$. Consider the preference relation $\succ$:

$$\{\theta_1,\theta_2\} \succ \{\theta_1\}, \{\theta_1,\theta_2\} \succ \{\theta_2\}, \{\theta_1,\theta_2\} \succ \emptyset,$$

$$\{\theta_1\} \succ \{\theta_2\}, \{\theta_1\} \succ \emptyset, \{\theta_2\} \succ \{\theta_1\}, \{\theta_2\} \succ \emptyset.$$

Obviously, this preference relation satisfies axioms B2)–B5) but not axiom B1), because $\{\theta_1\} \succ \{\theta_2\}$ and$\{\theta_2\} \succ \{\theta_1\}$. That is, axiom B1) is independent of other axioms.

Let $\Theta = \{\theta_1, \theta_2\}$. Consider the preference relation:

$$\{\theta_1,\theta_2\} \succ \{\theta_1\}, \{\theta_1,\theta_2\} \succ \emptyset, \{\theta_2\} \succ \emptyset.$$

This preference relation satisfies all the axioms except B2), because $\neg(\{\theta_2 \succ \theta_1\})$ and $\neg(\{\theta_1\} \succ \emptyset)$ but $\{\theta_2\} \succ \emptyset$.

Let $\Theta = \{\theta_1, \theta_2, \theta_3\}$. The preference relation:

$$\{\theta_1,\theta_2,\theta_3\} \succ \{\theta_2\} \succ \{\theta_2,\theta_3\} \succ \begin{matrix} \{\theta_1,\theta_2\} \\ \{\theta_1,\theta_3\} \end{matrix} \succ \begin{matrix} \{\theta_1\} \\ \{\theta_3\} \\ \emptyset \end{matrix}$$

satisfies all the axioms except B3), because $\{\theta_2\} \succ \{\theta_2, \theta_3\}$.

Let $\Theta = \{\theta_1, \theta_2, \theta_3\}$. The preference relation:

$$\begin{matrix} \{\theta_1,\theta_2,\theta_3\} \\ \{\theta_1,\theta_2\} \\ \{\theta_1,\theta_3\} \end{matrix} \succ \{\theta_1\} \succ \begin{matrix} \{\theta_2,\theta_3\} \\ \{\theta_2\} \\ \{\theta_3\} \\ \emptyset \end{matrix}$$

satisfies all the axioms except B4), because $\{\theta_1, \theta_2\} \succ \{\theta_1\}$ but $\neg(\{\theta_1,\theta_2,\theta_3\} \succ \{\theta_1,\theta_3\})$.

Consider the preference relation $\succ$ such that$A \sim B$ for all $A, B \in 2^{\Theta}$. Obviously, this relation $\succ$ satisfies all the axioms except B5), because $\neg(\Theta \succ \emptyset)$.

Before presenting the main theorem for the existence of a belief function that fully agrees with a preference relation, we first derive a few useful lemmas.

Lemma 1: Let $\succ$be a preference relation satisfying axiom B1), i.e., $\succ$ is asymmetric. Then, for $A, B \in 2^{\Theta}$, $A \succeq B \Longleftrightarrow \neg(B \succ A)$.

Proof: By definition of $\succeq$,$A \succeq B \Longleftrightarrow (A \succ B$ or $A \sim B)$. If $A \succ B$, then from axiom Q1) we obtain $\neg(B \succ A)$; if $A \sim B$, then by definition of $\sim$, $\neg(B \succ A)$ holds. Therefore, $A \succeq B \Longrightarrow \neg(B \succ A)$. Now suppose $\neg(B \succ A)$ holds. This means that we have either $A \succ B$ or $\neg(A \succ B)$. Thus, we can conclude that $A \succ B$ or $A \sim B$ holds. That is, $\neg(B \succ A) \Longrightarrow A \succeq B$.

Lemma 1 says that if axiom B1) holds for $\succ$, then$A \succeq B$ and $\neg(B \succ A)$ are equivalent statements. In this case, the dominance axiom B3) can be equivalently expressed as $A \supseteq B \Longrightarrow A \succeq B$.

Lemma 2: Let $\succ$ be a preference relation satisfying axioms B1) and B2), i.e., $\succ$ is asymmetric and negatively transitive. Then the binary relation $\sim$ defined by $A \sim B \Longleftrightarrow (\neg(A \succ B), \neg(B \succ A))$ is an equivalence relation on 2^{Θ}. Moreover, if Θ is finite, the power set 2^{Θ} can be partitioned into the equivalence classes $E_0, \ldots, E_k$ of n

The proof that the relation $\sim$ is an equivalence relation can be found in Fishburn [8, Theorem 2.1(c), p. 12]. The second part of Lemma 2 follows from a theorem given by Roberts [15, Theorem 3.1, page 101], which can also be found in Wong *et al.* [25].

Lemma 3: Let $\succ$ be a preference relation satisfying axioms B1), B3), and B4). If $\theta \notin A$ and $A \cup \{\theta\} \sim A$, then for any proper subset $B \subset A$, $B \cup \{\theta\} \sim B$.

Proof: Let $C = A - B$. Since $A \supset B$, we have $A = B \cup C$ and $B \cap C = \emptyset$. Obviously, $B \cup \{\theta\} \supseteq B$. By the dominance axiom B3), $B \cup \{\theta\} \succeq B$. Suppose $B \cup \{\theta\} \succ B$. Since $(B \cup \{\theta\}) \cap C = \emptyset$ and, by the axiom of partial monotonicity B4), we obtain $A \cup \{\theta\} \succ A$. This contradicts the assumption that $A \cup \{\theta\} \sim A$. Therefore, $B \cup \{\theta\} \sim B$.Q.E.D.

Theorem 4: Let Θ be a frame and $\succ$ a preference relation defined on 2^Θ. There exists a belief function, Bel: $2^\Theta \to [0,1]$, satisfying: for $A, B \in 2^\Theta$,

$$A \succ B \Longleftrightarrow \mathrm{Bel}(A) > \mathrm{Bel}(B),$$

if and only if the preference relation $\succ$ satisfies axioms B1)–B5).

Proof: (*only if*). Suppose there exists a belief function Bel:$2^\Theta \to [0,1]$ such that $A \succ B \Longleftrightarrow \mathrm{Bel}(A) > \mathrm{Bel}(B)$. The asymmetric and negatively transitive properties of $\succ$ immediately follow from the properties of $>$ on real numbers. In other words, axioms B1) and B2) hold. Now assume that $A \supseteq B$. Then there exists a $C \in 2^\Theta$ with $A = B \cup C$ and $B \cap C = \emptyset$. Hence, $\mathrm{Bel}(A) = \mathrm{Bel}(B \cup C) \geq \mathrm{Bel}(B) + \mathrm{Bel}(C) \geq \mathrm{Bel}(B)$, namely, $\neg(\mathrm{Bel}(B) > \mathrm{Bel}(A))$. We can therefore conclude that $A \supseteq B \Longrightarrow \neg(B \succ A)$. This means that the dominance axiom B3) holds. Assume that $A \supset B$, $A \cap C = \emptyset$, and $A \succ B$. We have $A \cup (B \cup C) = A \cup C$, $A \cap (B \cup C) = B$, and $\mathrm{Bel}(A) > \mathrm{Bel}(B)$. It follows:

$$\begin{aligned}\mathrm{Bel}(A \cup C) &= \mathrm{Bel}(A \cup (B \cup C)) \\ &\geq \mathrm{Bel}(A) + \mathrm{Bel}(B \cup C) - \mathrm{Bel}(A \cap (B \cup C)) \\ &= \mathrm{Bel}(B \cup C) + [\mathrm{Bel}(A) - \mathrm{Bel}(B)] \\ &> \mathrm{Bel}(B \cup C).\end{aligned}$$

Hence, $A \cup C \succ B \cup C$. That is, axiom B4) holds. Axiom B5) can be trivially proved from the fact that $\mathrm{Bel}(\Theta) = 1 > 0 = \mathrm{Bel}(\emptyset)$.

(*if*). Suppose axioms B1)–B5) hold for $\succ$. Our objective is to show that there exists a belief function, Bel : $2^\Theta \to [0,1]$, such that for $A, B \in 2^\Theta$, $A \succ B \Longleftrightarrow \mathrm{Bel}(A) > \mathrm{Bel}(B)$, i.e., Bel fully agrees with $\succ$.

If axioms B1) and B2) hold for a preference relation, Lemma 2 states that $\sim$ is an equivalence relation. Moreover, since axiom B5) holds, based on the relation $\sim$, we can partition 2^Θ into at least two equivalence classes $E_0, \ldots, E_k$ ($k \geq 1$) such that for $A \in E_i$ and $B \in E_j$, $A \succ B \Longleftrightarrow i > j$. Note that $\emptyset \in E_0$ and $\Theta \in E_k$. In what follows, we will denote an equivalence class by either E_i or $[A]$ if $A \in E_i$. For example, E_0 may be written as $[\emptyset]$ and E_k as $[\Theta]$.

One may recursively define a function f on the equivalence classes as:

$$(1)\ f(E_0) = 0,$$
$$(2)\ f(E_{n+1}) = \max\{f'(E_{n+1}), f(E_n) + 1\}$$

where

$$f'(E_{n+1}) = \max_{A \in E_{n+1}} \{-\sum_{A \supset B} (-1)^{|A-B|} f([B])\}$$

if for every $A \in E_{n+1}$, $A \supset B \Longrightarrow B \notin E_{n+1}$; otherwise $f'(E_{n+1}) = f(E_n) + 1$. The symbol $|\cdot|$ denotes the cardinality of a set. First we want to show that $A \succ B \Longleftrightarrow f([A]) > f([B])$. Suppose $A \succ B$ is true. Then we have $A \in E_i$ and $B \in E_j$ with $i > j$. By definition, $f([A]) \geq f(E_i) \geq f(E_j) + 1 = f([B]) + 1 > f([B])$. That is, $A \succ B \Longrightarrow f([A]) > f([B])$. Now assume that $f([A]) > f([B])$ is true. Then A and B cannot be in the same equivalence class; otherwise, according to the definition of f, we would have $f([A]) = f([B])$. Thus, either $A \succ B$ or $B \succ A$ must be true. Suppose $B \succ A$ is true, i.e., $f([B]) > f([A])$. This is a contradiction. Thus, $A \succ B$ is true. We can therefore conclude that $A \succ B \Longleftrightarrow f([A]) > f([B])$.

Based on the function f, one can define another function, $m' : 2^\Theta \to [0,1]$, as:

$$\begin{aligned}m'(A) &= f([A]) + \sum_{A \supset B} (-1)^{|A-B|} f([B]) \\ &= \sum_{A \supseteq B} (-1)^{|A-B|} f([B]).\end{aligned}$$

Obviously, $m'(\emptyset) = 0$. Before proving $m'(A) \geq 0$ for every $A \in 2^\Theta$, let us first show that if $A \supset B$ and $f([A]) = f([B])$, then $m'(A) = 0$. Let $D = A - B$, $A = B \cup D$, and $B \cap D = \emptyset$. We will prove the desired result for $|D| = 1$ and $|D| > 1$ separately. Suppose $|D| = 1$. Let $A = B \cup \{\theta\}$, and $\theta \notin B$. By the assumption $f([A]) = f([B])$, we obtain $f([B \cup \{\theta\}]) = f([B])$. By the definition of m'

$$\begin{aligned}m'(B \cup \{\theta\}) &= f([B \cup \{\theta\}]) + \sum_{B \cup \{\theta\} \supset C} (-1)^{|B \cup \{\theta\} - C|} f([C]) \\ &= f([B \cup \{\theta\}]) + (-1)^{|B \cup \{\theta\} - B|} f([B]) + \\ &\quad \sum_{B \supset C} \{(-1)^{|B \cup \{\theta\} - C|} f([C]) \\ &\quad + (-1)^{|B \cup \{\theta\} - C \cup \{\theta\}|} f([C \cup \{\theta\}])\}.\end{aligned}$$

Since $f([B \cup \{\theta\}]) = f([B])$, $B \cup \{\theta\} \sim B$. By Lemma 3, we have $C \cup \{\theta\} \sim C$ for every $C \subset B$. Hence, $f([C]) = f([C \cup \{\theta\}])$. It follows:

$$\sum_{B \supset C} \left\{ (-1)^{|B \cup \{\theta\} - C|} f([C]) + (-1)^{|B \cup \{\theta\} - C \cup \{\theta\}|} f([C \cup \{\theta\}]) \right\} = 0.$$

Finally, from $f([A]) = f([B]) = f([B \cup \{\theta\}])$, we obtain:

$$m'(A) = f([B \cup \{\theta\}]) + (-1)^{|B \cup \{\theta\} - B|} f([B]) = 0.$$

Now if $|D| > 1$, then D can be expressed as $D = D' \cup \{\theta\}$, $D' \neq \emptyset$, and $A = B \cup D' \cup \{\theta\}$. We have

$$A = B \cup D \supseteq B \cup D' \supseteq B.$$

By the dominance axiom B3),

$$B \cup D \succeq B \cup D' \succeq B.$$

That is,

$$f([B \cup D]) \geq f([B \cup D']) \geq f([B]).$$

By assumption, $f([A]) = f([B \cup D]) = f([B])$. Thus, $f([B \cup D]) = f([B \cup D'])$. That is, $f([A]) = f([B \cup D'])$. Let $B' = B \cup D'$. We have $A \supset B \cup D' = B'$, $f([A]) = f([B'])$, $A = B' \cup \{\theta\}$, and $\theta \notin B'$. The problem is then reduced to the previous case in which D contains only a single element. Hence, $m'(A) = 0$.

From the previous analysis, we know that for any $A \in 2^\Theta$, if there exists a $B \subset A$ with $f([B]) = f([A])$, then $m'(A) = 0$; otherwise,

$$m'(A) = f([A]) + \sum_{A \supset B} (-1)^{|A-B|} f([B]).$$

On the other hand, by the definition of f we have

$$f([A]) \geq - \sum_{A \supset B} (-1)^{|A-B|} f([B]).$$

Thus, $m'(A) \geq 0$ for every $A \in 2^\Theta$.

Based on the Möbius inversion, for every $A \in 2^\Theta$:

$$f([A]) = \sum_{A \supseteq B} m'(B).$$

and, $f([\Theta]) = \sum_{\Theta \supseteq B} m'(B) > 0$. Since $\Theta \succ \emptyset \Leftrightarrow f([\Theta])$ and by construction of $f([\Theta]) = f(E_o) = 0, f([\Theta])$. Now let $m(A) = m'(A)/f([\Theta])$. This implies that $\sum_{\Theta \supseteq B} m(B) = 1$. The function m is therefore a basic probability assignment. According to axioms M1)–M3), the function $\mathrm{Bel}(A) = \sum_{A \supseteq B} m(B)$ is a belief function. Moreover,

$$\begin{aligned}\mathrm{Bel}(A) &= \sum_{A \supseteq B} m(B) \\ &= \sum_{A \supseteq B} \frac{m'(B)}{f([\Theta])} = \frac{1}{f([\Theta])} \sum_{A \supseteq B} m'(B) = \frac{f([A])}{f([\Theta])}.\end{aligned}$$

Since $A \succ B \iff f([A]) > f([B])$, we can immediately conclude that $A \succ B \iff \mathrm{Bel}(A) > \mathrm{Bel}(B)$.

The following example illustrates how to construct a belief function for a belief relation.

Example 2: Let $\Theta = \{\theta_1, \theta_2, \theta_3\}$. Consider a preference relation $\succ$ on 2^Θ defined by:

$$\{\theta_1, \theta_2, \theta_3\} \succ \begin{matrix} \{\theta_1, \theta_3\} \\ \{\theta_2, \theta_3\} \end{matrix} \succ \begin{matrix} \{\theta_1\} \\ \{\theta_1, \theta_2\} \end{matrix} \succ \begin{matrix} \emptyset \\ \{\theta_2\} \\ \{\theta_3\} \end{matrix}$$

This preference relation $\succ$ indeed satisfies axioms B1)–B5); hence, it is a belief relation. The corresponding relation $\sim$ divides 2^Θ into four equivalence classes: $E_0 = [\emptyset, \{\theta_2\}, \{\theta_3\}]$, $E_1 = [\{\theta_1\}, \{\theta_1, \theta_2\}]$,
$E_2 = [\{\theta_1, \theta_3\}, \{\theta_2, \theta_3\}]$, and $E_3 = [\{\theta_1, \theta_2, \theta_3\}]$. By Theorem 4, we know that there exists a belief function that fully agrees with $\succ$.

First, we construct afunction f based on formulas 1) and 2):

E_0: According to formula 1), let $f(E_0) = 0$.

E_1: Since $\{\theta_1, \theta_2\} \supset \{\theta_1\}$, let $f'(E_1) = f(E_0) + 1 = 1$. Thus, $f(E_1) = \max\{1, f(E_0) + 1)\} = \max\{1, 1\} = 1$. E_2:

Note that for every $A \in E_2$, $A \supset B \implies B \notin E_2$.Thus, according to formula 2), we compute the value of $-\sum_{A \supset B} (-1)^{|A-B|} f([B])$ for every $A \in E_2$ as follows:

$$\begin{aligned} - \sum_{\{\theta_1,\theta_3\} \supset B} (-1)^{|\{\theta_1,\theta_3\}-B|} f([B]) &= f([\{\theta_1\}]) + f([\{\theta_3\}]) \\ &\quad - f([\emptyset]) \\ &= f(E_1) + f(E_0) \\ &\quad - f(E_0) \\ &= 1 + 0 - 0 = 1, \\ - \sum_{\{\theta_2,\theta_3\} \supset B} (-1)^{|\{\theta_2,\theta_3\}-B|} f([B]) &= f([\{\theta_2\}]) + f([\{\theta_3\}]) \\ &\quad - f([\emptyset]) \\ &= f(E_0) + f(E_0) - f(E_0) \\ &= 0 + 0 - 0 = 0. \end{aligned}$$

From these values, we obtain $f'(E_2) = \max\{1, 0\} = 1$. Hence,

$$f(E_2) = \max\{f'(E_2), f(E_1) + 1\} = \max\{1, 2\} = 2$$

E_3: Since E_3 has only one element, we have

$$\begin{aligned} f'(E_3) = &- \sum_{\{\theta_1,\theta_2,\theta_3\} \supset B} (-1)^{|\{\theta_1,\theta_2,\theta_3\}-B|} f([B]) \\ =& f([\{\theta_1, \theta_2\}]) + f([\{\theta_1, \theta_3\}]) + f([\{\theta_2, \theta_3\}]) - \\ & f([\{\theta_1\}]) - f([\{\theta_2\}]) - f([\{\theta_3\}]) + f([\emptyset]) \\ =& f(E_1) + f(E_2) + f(E_2) - f(E_1) - f(E_0) - \\ & f(E_0) + f(E_0) \\ =& 1 + 2 + 2 - 1 - 0 - 0 + 0 = 4. \end{aligned}$$

This implies that $f(E_3) = \max\{f'(E_2), f(E_2) + 1)\} = \max\{4, 3\} = 4$.

We can now compute the values of thefunction m' for every $A \in 2^\Theta$ based on the formula $m'(A) =$

$\sum_{A \supseteq B}(-1)^{|A-B|} f([B])$:

$$
\begin{aligned}
m'(\emptyset) &= f([\emptyset]) = f(E_0) = 0,\\
m'(\theta_1) &= f([\{\theta_1\}]) - f([\emptyset]) = f(E_1) - f(E_0)\\
&= 1,\\
m'(\theta_2) &= f([\{\theta_2\}]) - f([\emptyset]) = f(E_0) - f(E_0)\\
&= 0,\\
m'(\theta_3) &= f([\{\theta_3\}]) - f([\emptyset]) = f(E_0) - f(E_0)\\
&= 0,\\
m'(\{\theta_1, \theta_2\}) &= f([\{\theta_1, \theta_2\}]) - f([\{\theta_1\}]) - f([\{\theta_2\}])\\
&\quad + f([\emptyset])\\
&= f(E_1) - f(E_1) - f(E_0) + f(E_0) = 0,\\
m'(\{\theta_1, \theta_3\}) &= f([\{\theta_1, \theta_3\}]) - f([\{\theta_1\}]) - f([\{\theta_3\}])\\
&\quad + f([\emptyset])\\
&= f(E_2) - f(E_1) - f(E_0) + f(E_0) = 1,\\
m'(\{\theta_2, \theta_3\}) &= f([\{\theta_2, \theta_3\}]) - f([\{\theta_2\}]) - f([\{\theta_3\}])\\
&\quad + f([\emptyset])\\
&= f(E_2) - f(E_0) - f(E_0) + f(E_0) = 2,\\
m'(\{\theta_1, \theta_2, \theta_3\}) &= f([\{\theta_1, \theta_2, \theta_3\}])\\
&\quad - f([\{\theta_1, \theta_2\}]) - f([\{\theta_1, \theta_3\}])\\
&\quad - f([\{\theta_2, \theta_3\}]) + f([\{\theta_1\}]) + f([\{\theta_2\}])\\
&\quad + f([\{\theta_3\}]) - f([\emptyset])\\
&= f(E_3) - f(E_1) - f(E_2) - f(E_2)\\
&\quad + f(E_1) + f(E_0) + f(E_0) - f(E_0)\\
&= 0.
\end{aligned}
$$

Since $f([\Theta]) = f(E_3) = 4$, the values of the basic probability assignment defined by $m(A) = m'(A)/f([\Theta])$ are:

$m(\{\theta_1, \theta_2, \theta_3\}) = 0;$
$m(\{\theta_1, \theta_3\}) = \frac{1}{4},$
$m(\{\theta_2, \theta_3\}) = \frac{2}{4};$
$m(\{\theta_1\}) = \frac{1}{4},$
$m(\{\theta_1, \theta_2\}) = 0;$
$m(\emptyset) = 0,$
$m(\{\theta_2\}) = 0,$
$m(\{\theta_3\}) = 0.$

Accordingly, the values of the belief function $\text{Bel}(A) = \sum_{A \supseteq B} m(B)$ are:

$\text{Bel}(\{\theta_1, \theta_2, \theta_3\}) = 1;$
$\text{Bel}(\{\theta_1, \theta_3\}) = \frac{2}{4},$
$\text{Bel}(\{\theta_2, \theta_3\}) = \frac{2}{4};$
$\text{Bel}(\{\theta_1\}) = \frac{1}{4},$
$\text{Bel}(\{\theta_1, \theta_2\}) = \frac{1}{4};$
$\text{Bel}(\emptyset) = 0,$
$\text{Bel}(\{\theta_2\}) = 0,$
$\text{Bel}(\{\theta_3\}) = 0.$

One can easily verify that such a belief function indeed fully agrees with the given belief relation.

The two representation theorems (Theorems 1 and 4) we have presented form a basis for the study of qualitative belief. A preference relation is a weak belief relation if there exists a belief function Bel satisfying the *single* implication, $A \succ B \Longrightarrow \text{Bel}(A) > \text{Bel}(B)$. It is a strict belief relation if there exists a belief function Bel satisfying the *double* implication, $A \succ B \Longleftrightarrow \text{Bel}(A) > \text{Bel}(B)$. Theorem 1 shows that axioms B1)–B3) are sufficient but not necessary conditions for the existence of a belief function partially agreeing with a preference relation. These three axioms characterize only a subset of the weak belief relations. That is, any preference relation satisfying axioms B1)–B3) is a weak belief relation, but not every weak belief relation satisfies axioms B1)–B3). In contrast, Theorem 4 demonstrates that axioms B1)–B5) are indeed both necessary and sufficient conditions for the existence of a belief function that fully agrees with a preference relation. These five axioms uniquely describe the set of strict belief relations. It should perhaps be mentioned that there exist other functions besides belief functions that are consistent with the strict belief relations.

In the following section, we demonstrate the close relationship between qualitative belief and probability based on our characterization of belief relations.

IV. Qualitative Belief Induced by Qualitative Probability

As already mentioned, belief is a generalization of probability. In the compatibility view, a belief function can be constructed from a probability function defined on the evidence frame [5], [19]. A natural question to ask is whether a similar relationship can be established between qualitative belief and qualitative probability. Since qualitative probability is based on Savage's seven postulates of a personalistic theory of decision [17], a positive answer to this question will provide additional arguments in favor of our axiomatization of qualitative belief.

Belief may be interpreted according to either the *allocation* or the *compatibility* view [12]. In the quantitative approach, the allocation view of belief functions provides a generalization of the Bayesian theory by distributing probability mass to propositions that are not necessarily singleton sets [20], [21], [23]. The basic probability assignment $m(A)$ can be regarded as the probability mass being assigned to set A but not to any subset of A. This view is particularly useful in the situation where the evidence is too vague to be described in terms of propositions. On the other hand, when there is enough information to formulate the evidence frame, to construct the compatibility relation between the evidence frame and the frame of interest, and to define the underlying probability function on the evidence frame, one may adopt the compatibility view to construct a belief function [19]. In this case, the resulting belief function corresponds to the lower probability function, whereas the plausibility function corresponds to the upper probability function.

In the qualitative approach, the allocation view demands that the belief relation be expressed directly on the propositions of interest. However, when one adopts the compatibility view, a belief relation can be constructed from a probability relation defined on the evidence frame.

Definition 4: Consider two frames T and Θ. An element $t \in T$ is *compatible* with an element $\theta \in \Theta$, written $tC\theta$, if the proposition $\{t\}$ does not contradict the proposition $\{\theta\}$.

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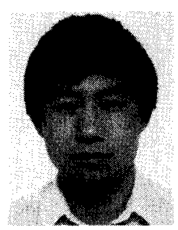

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P. Bollman, photograph and biography not available at the time of publication.

H. C. Bürger, photograph and biography not available at the time of publication.

probability relation $\succ_p$ on 2^T, the lower and upper probability relations, $_*\succ_p$ and $^*\succ_p$, on 2^Θ are defined as: for $A, B \in 2^\Theta$,

$$A \;_*\succ_p B \Longleftrightarrow \underline{\omega}(A) \succ_p \underline{\omega}(B),$$

and

$$A \;^*\succ_p B \Longleftrightarrow \overline{\omega}(A) \succ_p \overline{\omega}(B).$$

The following theorem shows that the lower and upper probability relations defined previously are in fact belief and plausibility relations.

Theorem 5: Let T and Θ be two frames, and let C be a compatibility relation between T and Θ. If a preference relation $\succ_p$ on 2^T satisfies axioms P1)–P4), the lower preference relation $_*\succ_p$ satisfies axioms B1)–B5). That is, $_*\succ_p$ is a belief relation. Furthermore, the upper preference relation $^*\succ_p$ is a plausibility relation, namely, $A \;^*\succ_p B \Longleftrightarrow B^c_* \succ_p A^c$.

Proof: Based on axioms P1) and P2),one can easily show that the relation $_*\succ_p$ satisfies axioms B1) and B2). Axiom B3) follows from properties I6) and II3). Axiom B5) follows from axiom P4) and properties I7) and I8).

Now we have to prove that the relation $_*\succ_p$ indeed satisfies axiom B4). Assume that $A \supset B$, $A \cap C = \emptyset$, and $A_*\succ_p B$. Obviously, $B \cap C = \emptyset$. Based on properties I1) and I8), we have $\underline{\omega}(A) \cap \underline{\omega}(C) = \underline{\omega}(A \cap C) = \emptyset$ and $\underline{\omega}(B) \cap \underline{\omega}(C) = \underline{\omega}(B \cap C) = \emptyset$. Since $A \;_*\succ_p B$, by definition, $\underline{\omega}(A) \succ_p \underline{\omega}(B)$. Therefore, we can immediately conclude from axiom P3) that $\underline{\omega}(A) \cup \underline{\omega}(C) \succ_p \underline{\omega}(B) \cup \underline{\omega}(C)$. Let $\underline{\omega}'(A \cup C) = \underline{\omega}(A \cup C) - \underline{\omega}(A) \cup \underline{\omega}(C)$ and $\underline{\omega}'(B \cup C) = \underline{\omega}(B \cup C) - \underline{\omega}(B) \cup \underline{\omega}(C)$. Then, $(\underline{\omega}(B) \cup \underline{\omega}(C)) \cap \underline{\omega}'(B \cup C) = \emptyset$. Since $A \supset B$ and $A \cap C = \emptyset$, it follows that $\underline{\omega}'(A \cup C) \supseteq \underline{\omega}'(B \cup C)$. Hence, $(\underline{\omega}(A) \cup \underline{\omega}(C)) \cap \underline{\omega}'(B \cup C) = \emptyset$. By applying axiom P3) again, we obtain $(\underline{\omega}(A) \cup \underline{\omega}(C)) \cup \underline{\omega}'(B \cup C) \succ_p (\underline{\omega}(B) \cup \underline{\omega}(C)) \cup \underline{\omega}'(B \cup C)$. This means that $\underline{\omega}(A) \cup \underline{\omega}(C) \cup \underline{\omega}'(B \cup C) \succ_p \underline{\omega}(B \cup C)$. Furthermore, from $\underline{\omega}'(A \cup C) \supseteq \underline{\omega}'(B \cup C)$, we obtain $\underline{\omega}(A \cup C) \supseteq \underline{\omega}(A) \cup \underline{\omega}(C) \cup \underline{\omega}'(B \cup C)$. By property II3), $\underline{\omega}(A \cup C) \succeq_p \underline{\omega}(A) \cup \underline{\omega}(C) \cup \underline{\omega}'(B \cup C)$. Finally, from property II2) we have $\underline{\omega}(A \cup C) \succ_p \underline{\omega}(B \cup C)$. Hence, $A \cup B \;_*\succ_p B \cup C$.

Since $_*\succ_p$ satisfies axioms B1)–B5), it is a belief relation. Suppose $B^c \;_*\succ_p A^c$. Then, $\underline{\omega}(B^c) \succ_p \underline{\omega}(A^c)$. From property I5), we obtain $(\overline{\omega}(B))^c \succ_p (\overline{\omega}(A))^c$. Property II4) implies that $\overline{\omega}(A) \succ_p \overline{\omega}(B)$. Thus, $A \;^*\succ_p B$. Similarly, we can show that if $A \;^*\succ_p B$, then $B^c \;_*\succ_p A^c$. That is, $^*\succ_p$ is a plausibility relation.

The example given below demonstrates the procedure for constructing belieffrom probability, quantitatively and qualitatively.

Example 3: Let $T = \{t_1, t_2\}$, and $\Theta = \{\theta_1, \theta_2, \theta_3\}$. Consider a compatibility relation between T and Θ:

$$t_1 C \theta_1, \quad t_2 C \theta_1, \quad t_2 C \theta_2, \quad t_2 C \theta_3.$$

Assume that a probability function P on 2^T is defined by $P(\{t_1\}) = 0.8$ and $P(\{t_2\}) = 0.2$. Based on definition 7 and the given compatibility relation, we can compute the lower and upper probabilities. For instance,

$$P_*(\{\theta_1\}) = P(\underline{\omega}(\{\theta_1\})) = P(\{t_1\}) = 0.8,$$
$$P^*(\{\theta_1\}) = P(\overline{\omega}(\{\theta_1\})) = P(\{t_1, t_2\}) = 1.0.$$

TABLE I
LOWER AND UPPER PREIMAGES AND PROBABILITY

A	$\underline{\omega}(A)$	$\overline{\omega}(A)$	$P_*(A)$	$P^*(A)$
$\emptyset$	$\emptyset$	$\emptyset$	0.0	0.0
$\{\theta_1\}$	$\{t_1\}$	$\{t_1, t_2\}$	0.8	1.0
$\{\theta_2\}$	$\emptyset$	$\{t_2\}$	0.0	0.2
$\{\theta_3\}$	$\emptyset$	$\{t_2\}$	0.0	0.2
$\{\theta_1, \theta_2\}$	$\{t_1\}$	$\{t_1, t_2\}$	0.8	1.0
$\{\theta_1, \theta_3\}$	$\{t_1\}$	$\{t_1, t_2\}$	0.8	1.0
$\{\theta_2, \theta_3\}$	$\emptyset$	$\{t_2\}$	0.0	0.2
$\{\theta_1, \theta_2, \theta_3\}$	$\{t_1, t_2\}$	$\{t_1, t_2\}$	1.0	1.0

The lower and upper probabilities for all subsets of Θ are given in Table I. It can be easily seen that P_* is a belief function, and $P^*(A) = 1 - P_*(A^c)$.

Suppose we have a probability relation on 2^T:

$$\{t_1, t_2\} \succ_p \{t_1\} \succ_p \{t_2\} \succ_p \emptyset.$$

Note that the probability function given earlier fully agrees with this probability relation. From this relation, we can construct the following lower and upper probability relations $_*\succ_p$ and $^*\succ_p$ on 2^Θ:

$$K\{\theta_1,\theta_2,\theta_3\} \;_*\succ_p \begin{matrix}\{\theta_1\}\\ \{\theta_1,\theta_2\}\\ \{\theta_1,\theta_3\}\end{matrix} \;_*\succ_p \begin{matrix}\emptyset\\ \{\theta_2\}\\ \{\theta_3\}\\ \{\theta_2,\theta_3\}\end{matrix}$$

and

$$\begin{matrix}\{\theta_1\}\\ \{\theta_1,\theta_2\}\\ \{\theta_1,\theta_3\}\\ \{\theta_1,\theta_2,\theta_3\}\end{matrix} \;^*\succ_p \begin{matrix}\{\theta_2\}\\ \{\theta_3\}\\ \{\theta_2,\theta_3\}\end{matrix} \;^*\succ_p \emptyset.$$

It can be easily verified that $_*\succ_p$ and $^*\succ_p$ are indeed belief and plausibility relations, P_* fully agrees with $_*\succ_p$, and P^* fully agrees with $^*\succ_p$.

V. CONCLUSION

In this paper we have studied the axiomatization of weak and strict belief relations. We have shown that the latter is consistent (compatible) with a belief function, while the former is not. The main results of this study are contained in two representation theorems (Theorems 1 and 4). The established relationship between qualitative belief and qualitative probability is interesting as it may lead to a better understanding and useful applications of belief structures in uncertainty management.

An important issue that remains controversial is the use of belief functions for decision making. One may develop an expected utility model for belief functions as Savage did in probability theory [9], [17]. On the other hand, one can first construct a probability function from a belief function and then apply the probabilistic utility model to make decisions [22]. The results presented here can be regarded as a first step toward developing a generalized utility theory for both probability and belief functions.

That is, the answer t to the question that defines T does not exclude the possibility that θ is the answer to the question that defines Θ.

Compatibility is symmetric: t is compatible with θ if and only if θ is compatible with t. A *compatibility relation* between two frames T and Θ is a subset of pairs (t, θ) in the Cartesian product $T \times \Theta$ such that $t\mathcal{C}\theta$. It is perhaps worth mentioning here that how one defines the frames T and Θ, the preference relation $\succ$ on 2^T, and the compatibility relation between the two frames, is relative to one's knowledge and opinion; hence, it is purely epistemic.

Definition 5: A compatibility relation$\mathcal{C}$ between two frames T and Θ is said to be *complete*, if for any $t \in T$ there exists at least one $\theta \in \Theta$ such that $t\mathcal{C}\theta$, and vice versa.

Hereafter, we will assume that all the compatibility relations between T and Θ are complete, because one can always obtain a reduced frame of T by deleting those elements in T that are not compatible with any element in Θ, and vice versa.

Definition 6: Given a compatibility relation $\mathcal{C}$ between T and Θ, one can define a mapping Γ that assigns a subset $\Gamma(t) \subseteq \Theta$ to every $t \in T$ as follows:

$$\Gamma(t) = \{\theta \in \Theta \mid t\mathcal{C}\theta\}.$$

Conversely, for any subset $A \subseteq \Theta$, one can define the lower and upper *preimages* of A, written $\underline{\omega}(A)$ and $\overline{\omega}(A)$, as:

$$\underline{\omega}(A) = \{t \in T \mid \Gamma(t) \subseteq A\},$$
$$\overline{\omega}(A) = \{t \in T \mid \Gamma(t) \cap A \neq \emptyset\}.$$

The set $\underline{\omega}(A)$ consists of all the elements in T that are compatible with only those elements in A, and the set $\overline{\omega}(A)$ consists of all the elements in T that are compatible with at least one element in A. For any subsets $A, B \subseteq \Theta$, the following properties hold [14], [18]:

I1) $\underline{\omega}(A \cap B) = \underline{\omega}(A) \cap \underline{\omega}(B)$,

I2) $\underline{\omega}(A \cup B) \supseteq \underline{\omega}(A) \cup \underline{\omega}(B)$,

I3) $\overline{\omega}(A \cap B) \subseteq \overline{\omega}(A) \cap \overline{\omega}(B)$,

I4) $\overline{\omega}(A \cup B) = \overline{\omega}(A) \cup \overline{\omega}(B)$,

I5) $\underline{\omega}(A^c) = (\overline{\omega}(A))^c, \quad \overline{\omega}(A^c) = (\underline{\omega}(A))^c$,

I6) $A \supseteq B \Longrightarrow (\underline{\omega}(A) \supseteq \underline{\omega}(B), \overline{\omega}(A) \supseteq \overline{\omega}(B))$,

I7) $\underline{\omega}(\Theta) = \overline{\omega}(\Theta) = T$,

I8) $\underline{\omega}(\emptyset) = \overline{\omega}(\emptyset) = \emptyset$.

Let P be a probability function defined on 2^T. Based on the lower and upper preimages of $A \in 2^\Theta$, Dempster [5] introduced the lower and upper probabilities of A, that are obtained by transferring the probability function P from frame T to frame Θ.

Definition 7: Let T and Θ be two frames, and let $\mathcal{C}$ be a complete compatibility relation between T and Θ. For a given probability function P defined on 2^T, the lower and upper probabilities, P_* and P^*, of an arbitrary set $A \in 2^\Theta$ are defined by:

$$P_*(A) = P(\underline{\omega}(A)),$$
$$P^*(A) = P(\overline{\omega}(A)).$$

The lower probability $P_*(A)$ is the smallest amount of probability transferable to A, whereas the upper probability $P^*(A)$ is the largest possible amount of probability transferable to A. The lower probability is a belief function satisfying Shafer's axioms, and the upper probability is the corresponding plausibility function. This can be easily seen from the fact that the function, $m : 2^\Theta \to [0,1]$, defined by:

$$m(F) = \sum_{\Gamma(t)=F} P(\{t\})$$

for all $F \in 2^\Theta$, is indeed a basic probability assignment. Moreover,

$$P_*(A) = P(\underline{\omega}(A)) = \sum_{t\in\underline{\omega}(A)} P(\{t\})$$
$$= \sum_{A\supseteq\Gamma(t)} P(\{t\}) = \sum_{A\supseteq B}\sum_{\Gamma(t)=B} P(\{t\}) = \sum_{A\supseteq B} m(B).$$

Based on property I5), it follows that $P^*(A) = 1 - P_*(A^c)$ is a plausibility function.

Similar to transferring a probability function, we can also transfer qualitative probability from a frame T to obtain qualitative belief in another frame Θ.

Definition 8: Let T be a frame. A preference relation $\succ_p$ defined on 2^T is called a *probability relation* or *qualitative probability* ([8], [17]) if for $X, Y, Z \in 2^T$,

P1) (asymmetry): $X \succ_p Y \Longrightarrow \neg(Y \succ_p X)$,

P2) (negative transitivity): $(\neg(X \succ_p Y), \neg(Y \succ_p Z)) \Longrightarrow \neg(X \succ_p Z)$,

P3) (monotonicity): $X \cap Z = Y \cap Z = \emptyset \Longrightarrow (X \succ_p Y \Longleftrightarrow X \cup Z \succ_p Y \cup Z)$,

P4) (improbability of impossibility and nontriviality):
$\neg(\emptyset \succ_p X), \ T \succ_p \emptyset$.

Axioms P1) and P2) are identical to axioms B1) and B2). Axioms P3) and P4) are considered to be essential for the characterization of a *more probable than* relation. Axioms P1)–P4) imply the following properties:

II1) $X \succeq_p Y \Longleftrightarrow \neg(Y \succ_p X)$,

II2) $(X \succeq_p Y, Y \succ_p Z) \Longrightarrow X \succ_p Z$,

II3) $X \supseteq Y \Longrightarrow \neg(Y \succ_p X)$,

II4) $X \succ_p Y \Longleftrightarrow Y^c \succ_p X^c$.

Property II1) is implied by axiom P1) as shown by Lemma 1. Property II2) follows from axioms P1) and P2) [25]. The property of dominance II3) is obtained from the improbability of impossibility axiom, $\neg(\emptyset \succ_p X)$, and the axiom of monotonicity P3). Based on property II1), property II3) can be expressed as $X \supseteq Y \Longrightarrow X \succeq_p Y$. Property II4) is implied by axiom P3). Comparing axioms B1)–B5) with P1)–P4), it is clear that qualitative probability is qualitative belief. In other words, qualitative belief can be interpreted as a generalization of qualitative probability.

It should be noted that axioms P1)–P4) are necessary but not sufficient conditions for the existence of a quantitative probability function satisfying Kolmogorov axioms and fully agreeing with $\succ_p$ [7], [17].

Definition 9: Let T and Θ be two frames, and let $\mathcal{C}$ be a compatibility relation between T and Θ. For a given