A Logic of Comparative Support: Qualitative Conditional Probability Relations Representable by Popper Functions

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Draft submitted to Alan Hájek and Chris Hitchcock, eds.,
Oxford Handbook of Probabilities and Philosophy, Oxford University Press.

1. Introduction

This article explicates a logic of comparative support relations of the form "conclusion C_1 is supported by premises P_1 at least as strongly as conclusion C_2 is supported by premises P_2", abbreviated "$C_1|P_1 \succcurlyeq C_2|P_2$". This logic is a generalization of the logic of comparative conditional probability first developed by B. O. Koopman (1940). The version of the logic on offer here contains several innovations. Its axioms will not presuppose deductive logic. Rather, classical logical entailment will fall out as a special case of comparative support. Furthermore, these comparative support relations turn out to be the qualitative conditional probabilistic analogs of the conditional probability functions known as *Popper functions*. As a result, this logic can capture all of the features of Bayesian confirmation theory. Thus, this *logic of comparative support* provides a foundation for the notion of evidential support that underlies both classical deductive logic and probabilistic inductive logic. Along the way I'll provide background on the nature of the *Popper functions*, including a particularly spare axiomatization of the *Popper functions* that won't presuppose deductive logic.

Let's begin with some background on logics of comparative probability. For some applications numerical probability values are unimportant, and may even seem inappropriate. In such cases the usefulness of probability functions may only derive from their ability to represent a *probabilistic ordering* among possible states of affairs – i.e. that some states of affairs are *more probable* (or *more plausible*) than others. In such cases the use of numerical probability functions may seem objectionable because the precise numerical values appear to be meaningless. This may give rise to philosophical objections to the use of probabilities in these contexts. *Comparative probability relations* provide a way to sidestep objections to precise numerical values. They provide the desired probabilistic orderings without drawing on numerical values.

The version of comparative probability relations on offer here will take *conditional probability* as basic. Unconditional probabilities may be represented by conditional probabilities in which the condition is a tautology. Just as many distinct conditional probability functions P satisfy the axioms of probability theory, there will be many distinct *comparative conditional probability* relations $\succcurlyeq$ that satisfy the axioms I'll provide for *comparative conditional probability relations*. Each *comparative conditional probability relation* is a *weak partial order* on pairs of sentences (i.e. it is a *transitive* and *reflexive* relation). Each such relation takes the form $A|B \succcurlyeq C|D$, which may be read to say, "A given B *is at least as probable as* C given D". Furthermore, each relation $\succcurlyeq$ that satisfies the following axioms will turn out to correspond to a kind of numerical conditional probability function called a *Popper function*. *Popper functions* are numerical probability functions that take conditional probabilities as basic, rather than defining conditional probability in terms of unconditional probability. It will turn out that for each numerical *Popper function* P, there is a corresponding *comparative conditional probability relation* $\succcurlyeq$ (satisfying the following axioms) that agrees with P – i.e. for all sentences A, B, C, D, whenever the probabilistic relationship P[A | B] ≥ P[C | D] holds, the corresponding qualitative relationship A|B $\succcurlyeq$ C|D holds. Furthermore, it turns out that each comparative conditional probability relation $\succcurlyeq$ that satisfies our axioms can be *represented* by some numerical *Popper function* that provides the same comparative probability ordering among possible states of affairs. That is, a representation theorem will establish that for each comparative conditional probability relation $\succcurlyeq$ that satisfies (a version of) our axioms, there is a corresponding Popper function P such that for all sentences A, B, C, D, whenever the qualitative relationship A|B $\succcurlyeq$ C|D holds, the corresponding probabilistic relationship P[A | B] ≥ P[C | D] holds. Thus, *comparative probability relations* are robust enough to provide all of the structure engendered by numerical probability functions in placing orderings among possible states of affairs. Indeed, we will find that some kinds of *comparative probability relations* can represent probabilistic ordering structures of a non-Archimedean kind not captured by numerical probability functions.

The axiomatic system I'll present is purely formal, so the notion of *comparative probability* involved may be interpreted in any of the various ways that numerical probability functions are interpreted. However, historically the notion of *comparative conditional probability* has been most closely associated with the notion of *comparative evidential support*. Moreover, it will be easier to motivate the proposed axioms if we give the comparative relations $\succcurlyeq$ some specific interpretive reading. So, in this article I will treat the qualitative relations as representations of *comparative evidential support relations*, understood as a kind of *comparative argument strength*. Thus, each such qualitative relationship 'A|B $\succcurlyeq$ C|D' will be read to say, "conclusion A is supported by the conjunction of premises B *at least as strongly as* conclusion C is supported by the conjunction of premises D". However, since the axiomatic treatment itself is purely formal, these same probabilistic relations may also be interpreted to represent other conditional probabilistic concepts, provided that conditional probability function (of the kind known as *Popper functions*) may also be so interpreted. For example, perhaps 'A|B $\succcurlyeq$ C|D' may also be interpreted to represent some notion of *comparative objective chance*: "for systems in state α, the chance of outcome A among those systems with attribute B *is at least as great as* the chance of outcome C among those with attribute D". Then the following representation theorems show that this non-numerical notion of *comparative chance* adequately captures all of the comparative probabilistic features embodied by numerical conditional chance functions. Under such an interpretation the logic explicated here provides a completely adequate treatment of *chance without numbers*.

Here is a bit of historical background to the logic of comparative probability relations of the sort explicated here. In his *Treatise on Probability* (1921) John Maynard Keynes develops a logic of evidential support that takes conditional probability functions as basic measures of argumentative support strength. The idea is that for some appropriate conditional probability function P, the conditional probability statement 'P[H | E] = *r*' means that the conjunction of premises E supports conclusion H with strength *r*. On Keynes' account the specific numerical values for support strengths are often of little importance, and indeed the "probability values" *r* assigned as the argument strengths for conditional probabilities P[H | E] need not be numbers at all. Rather, the Keynesian idea is that these "probabilities" are merely some partially ordered objects that are used to provide the partial ordering relationships among arguments in accord with their strengths. These partially ordered objects may be real numbers in some cases, but in other cases they may not be numbers at all – indeed, Keynes argues that in some cases they *cannot* be numbers. Whenever $P[H_1 \mid E_1] > P[H_2 \mid E_2]$, the argument from premises E_1 to conclusion H_1 is taken to be stronger than the argument from premises E_2 to conclusion H_2. The strongest arguments are the deductively valid ones – whenever $E_1 \models H_1$ (i.e. whenever E_1 deductively entails H_1), then no argument from premises E_2 to conclusion H_2 can be any stronger, so $P[H_1 \mid E_1] \geq P[H_2 \mid E_2]$. In that case the argument from E_1 to H_1 gets probability 1. Furthermore, Keynes holds that the comparative strengths of some argument pairs may be objectively *incomparable*. That is, Keynes argues that for some pairs of arguments $H_1|E_1$ and $H_2|E_2$, neither $P[H_1 \mid E_1] \geq P[H_2 \mid E_2]$ nor $P[H_2 \mid E_2] \geq P[H_1 \mid E_1]$.

Taking his cue from Keynes, the mathematician Bernard Osgood Koopman developed a purely qualitative account of conditional probability (1940). Koopman axiomatized comparative conditional probabilistic relations $\succcurlyeq$ with the goal of providing a more comprehensive account of Keynes' logic of comparative argument strength. Since then a number of researchers have proposed various axiomatizations of comparative probabilistic relations. Indeed, even before Koopman's work de Finetti (1937) had proposed qualitative comparative axioms for unconditional probabilities, where the basic notion $A \succcurlyeq B$ represents "A is at least as probable as B". Most approaches (for both unconditional and conditional probabilities) have supposed that the primitive relation $\succcurlyeq$ is a *complete order* – i.e. in the conditional-probability case it is usually assumed that for all A, B, C, and D, either $A|B \succcurlyeq C|D$ or $C|D \succcurlyeq A|B$. Furthermore, none of the systems of qualitative conditional probabilities developed thus far correspond to the most general kind of conditional probability functions now available, the *Popper functions*. The version of qualitative conditional probability on offer in this article is more general than other accounts in both of these respects. The axioms will permit the comparative relations to be merely partial orders, and the comparative relations that satisfy the axioms below will correspond to the full range of those conditional probability functions known as *Popper functions*. Indeed, the comparative probabilistic relations that satisfy one version of the following axioms will turn out to be representationally richer than the quantitative conditional probability functions.[1]

2. The *Popper-Field Conditional Probability Functions*

In this section I'll specify a version of the axioms for *Popper functions*. This kind of conditional probability function is of interest in its own right. All of the more usual kinds of conditional probability functions are themselves *Popper functions* – i.e. they satisfy the axioms for *Popper functions*. But also among the *Popper functions* are conditional probability functions that make important use of conditionalizations on statements that have probability 0. *Popper functions* are especially useful for our purposes because they can represent all of the non-numerical

[1] Because Koopman's work on comparative conditional probability relations axiomatizes them as partial orders, each such relation is representable by a set of precise conditional probability functions that covers a range (an interval) of probability values for one proposition given another. This provided a way into theories of imprecise and indeterminate probabilities. For a detailed account of theories of imprecise and indeterminate probabilities, see the article by Fabio Cozman in this volume.

comparative probability relations specified later. That is, for each non-numerical *comparative probability relation*, there is a *Popper function* that numerically yields the same comparative relationships; and every *Popper function* gives rise to a comparative relation that satisfies the axioms specified below for *comparative probability relations*.

To understand the relationship between *Popper functions* and *classical conditional probability functions*, think of it like this. Given any *unconditional* classical probability function P defined on sentences or propositions, *conditional probability* is usually *defined* as follows: whenever P[B] > 0, P[A | B] = P[(A·B)]/P[B]; and when P[B] = 0, P[A | B] is left undefined. Let's make a minor modification to this usual approach, and require instead that *classical conditional probability functions* make P[A | B] = 1 by default whenever P[B] = 0. Thus, on this approach *conditional probabilities* are always defined. Specified in this way, each *classical conditional probability function* is a simple kind of *Popper function*.

The class of all *Popper functions* is a little more general than this. For, among the *Popper functions* are probability functions that possess another feature that gives them a richer structure than the *classical conditional probability functions*. Here is how that more general feature works. For a given *Popper function* P, if we hold the condition statement B fixed, then the function P[| B] always behaves precisely like a classical probability function. However, whenever a statement C has 0 probability on B, P[C | B] = 0, the probability function resulting from P by now holding this conjunction (C·B) fixed, P[| (C·B)], may behave like an entirely different *classical probability function*. In general, a *Popper function* behaves like a ranked hierarchy of *classical probability functions*, where the transition from a classical probability function at one level in the hierarchy (the statement 'B' level) to a new classical probability function at a lower level in the hierarchy (the statement '(C·B)' level) is induced by conditionalization on a statement that has probability 0 at that higher (statement B) level. This provides a generalization of the usual conception of conditional probability.

This generalization of classical probability is useful for cases of the following sort. The probability that a randomly selected point will lie within a specific region of three-dimensional space described by A, given that it lies somewhere within a larger three-dimensional region described by B, may have some well-defined non-zero value – e.g. P[A | B] = 3/4. However, the probability that this same randomly selected point will lie precisely on the part of a plane described by C within the region described by B should presumably be 0, so we have 0 = P[C | B] = P[(A·C) | B]. However, given that this randomly selected point does indeed lie within the plane described by C within the region described by B, the probability that it lies within the region described by A (which, say, contains half of the plane described by C) may be perfectly well-defined: P[A | (C·B)] = 1/2. In addition, the probability that this randomly selected point will lie on the line segment described by D within the plane described by C should also presumably be 0, so we again have a situation where 0 = P[D | (C·B)] = P[(A·D) | (C·B)]. However, given that this randomly selected point does lie within the line segment described by D within the plane described by C within the region described by B, the probability that it lies in the region described by A (which, say, contains two-thirds of line segment described by D) may again be perfectly well-defined: P[A | (D·(C·B))] = 2/3. So, the general idea underlying the *Popper functions* is that a specific conditional probability function (a *Popper function*) may consist of a ranked hierarchy of classical probability functions, where conditionalizations on specific probability 0 statements at one level of the hierarchy can induce a transition to another perfectly good classical probability function defined at a lower level of the hierarchy.

One way that a logic of evidential support may employ this ranked structure is to place some sub-classes of alternative hypotheses at ranks below other alternatives. Alternative hypotheses at the lower rank only come into play as live options (having non-zero probability) when evidential events occur that have probability 0 (i.e. zero likelihoods) according to all alternative hypotheses at higher ranks. Such evidential events refute all of the higher ranked hypotheses, and thereby induce a shift to a new sub-class of alternative hypotheses that now rate as available (non-zero probability) options. Thus, the additional structure carried by *Popper functions* may provide a useful tool for modeling some aspects of hypothesis evaluation not captured by *classical conditional probability functions*.

The logic of the *Popper functions* has one other feature that may be particularly attractive to philosophical logicians. It provides a kind of *generalized logic of evidential support* that *need not* presuppose a pre-defined formal deductive logic. Rather, a relation that's equivalent to *classical deductive entailment* falls out of the probabilistic logic as a special case. Thus, a probabilistic logic of this sort is *autonomous* in that it need not pre-suppose classical deductive logic. It will turn out that the *logic of comparative conditional probability* also has this feature. It is a *generalized logic of evidential support* that need not presuppose a pre-defined formal deductive logic. Rather, a relation that's equivalent to *classical deductive entailment* will emerge from the *logic of comparative support* as a special case.

Various axiomatizations of the *Popper functions* are available. Following is an especially spare version.[2] This version does not presuppose a pre-defined notion of *logical entailment*, nor are *logically equivalent* sentences assumed to be inter-substitutable, nor are logically equivalent sentences assumed to have the same probability values. Rather, a notion of *logical entailment* that is equivalent to the *classical notion of logical entailment* is derivable from these axioms, and the substitutivity of logically equivalent sentences may be proved from these axioms. Thus, this approach to axiomatizing probability is *autonomous* with respect to classical deductive logic. Furthermore, these probability functions are only assumed to range over *some real numbers* – perhaps negative, and perhaps much greater than 1. Then it can be proved that all of these probabilities must range between 0 and 1.

> Let *L* be a language having the syntax of sentential logic. A *Popper function* is any function P from pairs of sentences of *L* to the real numbers (not necessarily restricted between 0 and 1) such that:[3]
> 0_P. for some E, F, G, H, $P[F \mid E] \neq P[G \mid H]$ (non-triviality)
> and for all sentences A, B, C,
> 1_P. $P[A \mid A] \geq P[B \mid B]$ (self-support)

[2] Suggested by a version in an appendix to (Popper, 1959).
[3] These axioms rely solely on *negation* and *conjunction*. Other logical terms for sentential logic (*or*, *if…then*, *if and only if*) may be treated as *defined terms* in the usual way: '(A∨B)' abbreviates '~(~A·~B)', '(A⊃B)' abbreviates '~(A·~B)', '(A≡B)' abbreviates '(~(A·~B)·~(~A·B))'.

2_P. P[A | C] $\geq$ P[(A·B) | C] (left simplification)
3_P. P[A | (B·C)] $\geq$ P[A | (C·B)] (right commutivity)
4_P. P[(A·B) | C] = P[A | (B·C)] $\times$ P[B | C] (conditionalization)
5_P. P[A | B] + P[~A | B] = P[B | B] unless P[D | B] = P[B | B] for all D (additivity).
(The subscript 'P' stands for "P*opper function*".)

This axiomatization shows that extremely weak conditions suffice to characterize the *Popper functions*. The following rules are then derivable as theorems (where 'B |= A' abbreviates "B *logically entails* A" in the classical deductive sense).[4] (The subscripts 'C' on the following rules stand for "Classical Logic", because most of these rules depend on some pre-defined notion of *classical logical entailment*.)

Let *L* be a language having the syntax of sentential logic. Each *Popper function* is a function P_α from pairs of sentences of *L* to the real numbers such that for all sentences A, B, and C:
0_C. if |= ~A and |= B (i.e. ~A is a contradiction and B is a tautology), then P[A | B] = 0
1_C. 1 $\geq$ P[A | B] $\geq$ 0
2_C. if B |= A, then P[A | B] = 1
3_C. if C |= B and B |= C, then P[A | B] = P[A | C]
4_C. P[(A·B) | C] = P[A | (B·C)] $\times$ P[B | C]
5_C. if C |= ~(A·B), then P[(A$\vee$B) | C] = P[A | C] + P[B | C] unless P[D | B] = 1 for all D.

The axioms 0_C-5_C are derivable from axioms 0_P-5_P, and *vice versa*. Thus, these latter rules can be used as an alternative (easier to use) axiomatization of the *Popper functions*. All of the usual classical theorems of probability theory are easily derived from these C-axioms. (E.g. it can be shown that logically equivalent sentences must have the same probability.)

Notice that the conditionalization axiom (4_C and 4_P) is more generous than is usually provided for by classical axiomatizations of probability. In cases where P[C | B] = 0 (so P[(A·C) | B] = 0 as well), the term P[A | (C·B)] is permitted to take on a value between 0 and 1, whereas on the usual axiomatizations P[A | (C·B)] would either remain undefined or be defined as equal to 1. This feature of conditionalization permits *Popper functions* to behave as sequentially ranked hierarchies of classical probability functions, where conditionalization on a sentence that has probability 0 at one rank drops the function down to a new classical probability function at a lower rank; and where that new function continues to behave classically unless conditionalized on some sentence that has probability 0 at that rank, in which case the function drops yet again to a new classical probability function at a yet lower rank; …, etc. Finally, the very bottom rank consists of logical contradictions, perhaps together with other sentences that "behave like logical contradictions" – where, to say that a sentence D "behaves like a logical contradiction" means that every sentence has probability 1 when conditionalized on D.

The *Popper functions* easily extend to a language *L* for predicate logic with identity, which is

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[4] This derivation draws on the following definition and theorem.
Definition: B *Popper-entails* A (abbreviated 'B $|=_P$ A') *just when*, for each *Popper function* P on *L*, for all C in P's language, P[A | C] $\geq$ P[B | C].
Theorem: B $|=_P$ A if and only if B |= A.

important if they are to be used as evidential support functions that apply to a language in which scientific hypotheses and theories are formulated. Hartry Field (1977) showed how to extend Popper's *autonomous axiomatization* (*autonomous* with regard to classical deductive logic) to predicate logic. Let's call the predicate logic version of the *Popper functions* the *Popper-Field functions*.

> Let L_0 be a language with the syntax for predicate logic (with identity), and consider a class of languages ***L*** that contains L_0 together with name-extensions of L_0 drawn from a countable set of new names – i.e., all languages in ***L*** are just like L_0 but have some finite or countably infinite number of names not contained in L_0. Define a *PF-class* to be a set of functions *M* such that each one of them, P_α, assigns real numbers (not necessarily between 0 and 1) to pairs of sentences of its language (which is one of the languages in ***L***) so as to satisfy axioms 0_P-5_P together with the following axioms:
> For all sentences C, formulas Fx with only x free, and names $c_1, ..., c_m$ in P_α's language,
> 6_P. $P_\alpha[((Fc_1 \cdot Fc_2) \cdot ... \cdot Fc_m) \mid B] \geq P_\alpha[\forall xFx \mid B]$
> 7_P. if $r > P_\alpha[\forall xFx \mid B]$, then there is a function P_β in *M* defined on a name extension of P_α's language that agrees with P_α on its language such that for some names $e_1, ..., e_n$ in P_β's language $r \geq P_\beta[((Fe_1 \cdot Fe_2) \cdot ... \cdot Fe_n) \mid B]$ (notice: this makes $P_\alpha[\forall xFx \mid B]$ the *greatest lower bound* of values of $P_\beta[((Fe_1 \cdot Fe_2) \cdot ... \cdot Fe_n) \mid B]$ for all the name extensions P_β of P_α in *M*).
> Now define the class ***PF*** of all *Popper-Field functions* on languages in ***L*** to be the union of all such *PF-classes M*.[5]

This is just a way of specifying how the quantifier "all" is supposed to work in a probabilistic logic, and of doing so in a way that does not presupposed classical deductive logic. Rule 6_P says that conjunctions of instances are always at least as probable as a universally quantified sentence. Intuitively this should hold because $\forall xFx$ should logically entail all instances of the form $((Fc_1 \cdot Fc_2) \cdot ... \cdot Fc_m)$, and entailed sentences should be at least as probable as the sentence that entails them.

The typical truth-value semantic for the universal quantifier $\forall$ says that $\forall xFx$ is true on a truth-value assignment T_α (i.e. $T_\alpha[\forall xFx]$ holds) *just in case* for every possible truth-value assignment T_β that's just like assignment T_α except that T_β extends T_α to truth-values involving additional new names, e_1, e_2, etc., (by applying those new names to additional objects in the object-domain of the interpretation), all of the sentences Fe_1, Fe_2, etc. turn out true on the extended assignment T_β (i.e. $T_\beta[Fe_i]$ holds for all of the new names e_i). Rule 7_p captures the same idea, but applied to probability functions P_α and their name extensions P_β (instead of truth-value assignments T_α and their name extensions T_β). The only difference is that for probability assignments we want some name extensions to suffice to bring the probabilities of the conjunctions, $((Fe_1 \cdot Fe_2) \cdot ... \cdot Fe_n)$, arbitrarily close to the probability of $\forall xFx$ for sufficiently large numbers of names (for sufficiently large values of n). Thus, we require that function P_α be extendable to functions P_β (among a class of such functions *M*) such that for any value of *r* greater than $P_\alpha[\forall xFx \mid B]$, there

[5] To include identity add the following two axioms:
8_P. $P_\alpha[\forall x\, x{=}x \mid B] \geq P_\alpha[B \mid B]$
9_P. $P_\alpha[Fb \mid ((Fa \cdot b{=}a) \cdot B)] \geq P_\alpha[B \mid B]$.

is some name extension P_β (among the name extensions of P_α in class *M*) such that for enough of its names we have $r \geq P_\beta[((Fe_1 \cdot Fe_2) \cdot ... \cdot Fe_n) \mid B] \geq P_\beta[\forall xFx \mid B] = P_\alpha[\forall xFx \mid B]$. So, in order to get the logic of the quantifiers to work in much the same way that it works for a truth-value semantics, we specify each *Popper-Field function* in terms of a class of related functions – a function P_α together with a so-called *PF-class M* of its name extensions. Then, finally, the class of all *Popper-Field functions* on a language (and its name extensions) for predicate logic *is* just the class ***PF*** that consists of all such functions P_α, including those functions P_β (in *PF-classes M*) that agree with them but also employ additional names.

One can prove that each of the above theorems 0_C-5_C holds for each *Popper-Field function* P_α (in the class *PF* of all such functions) on a language for predicate logic, where now the *logical entailment* relation, $\models$, in theorems 0_C-5_C is *logical entailment for predicate logic* (with identity). Furthermore, theorems 0_C-5_C suffice to prove each of the axioms for the *Popper-Field functions* (with the exception of axiom 7_P, which may be added to 0_C-5_C as a separate axiom, if desired). So, one may treat theorems 0_C-5_C as an alternative way to axiomatize the same collection of *Popper-Field functions* – a way that shows directly how the *Popper-Field functions* may be gotten by extending classical deductive logic.

Those who propose to use *Popper-Field functions* (or any other kinds of probability functions) as *confirmation functions* in a Bayesian logic of evidential support are often challenged to give an account of where the numerical *degrees of support* come from. What do these probabilistic numbers *mean* or *represent*? Subjectivist Bayesians may answer this question in terms of betting functions and Dutch book theorems – the idea being that confirmation functions are belief-strength functions, and that the numerical values of belief-strength functions represent ideally rational betting quotients. However, if a confirmation theorist holds the sort of logical account of confirmation functions endorsed by Keynes, an account wherein confirmation functions represent argument strengths, then another sort of answer to the "where do the numbers come from, and what do they represent?" question makes better sense – an answer via a representation theorem. On this approach the idea is that there is a deeper underlying *qualitative logic of comparative evidential support*, and the numerical probabilities provided by *Popper-Field functions* are merely a convenient representation of this deeper logic. I now proceed to specify the axioms for this deeper logic of *comparative support*. And I'll show how the *Popper-Field functions* represent this deeper qualitative logic of evidential support.[6]

3. Towards the Logic of *Comparative Evidential Support*: the *Proto-Support Relations*

A *comparative support relation* $\succcurlyeq$ is a relation among pairs of sentence pairs that satisfies axioms that seem appropriate to an intuitive conception of comparative argument strength. Associated with each relation $\succcurlyeq$ are several related relations that may be defined in terms of it. Here is a list of these related relations, together with their formal definitions in terms of the primitive support relation $\succcurlyeq$, and together with an appropriate informal reading of each.

[6] For an account of how the Popper Functions compare to other kinds of accounts of conditional probability functions, see the article by Kenny Easwaran in this volume.

A comparative support relation $\succcurlyeq$ is a relation of form $H_1|E_1 \succcurlyeq H_2|E_2$,
read "H_1 *is supported by* E_1 *at least as strongly as* H_2 *is supported by* E_2".

Define four associated relations as follows:

(1) $H_1|E_1 \succ H_2|E_2$ abbreviates "$H_1|E_1 \succcurlyeq H_2|E_2$ and not $H_2|E_2 \succcurlyeq H_1|E_1$",
read "H_1 *is supported by* E_1 *more strongly than* H_2 *is supported by* E_2";

(2) $H_1|E_1 \approx H_2|E_2$ abbreviates "$H_1|E_1 \succcurlyeq H_2|E_2$ and $H_2|E_2 \succcurlyeq H_1|E_1$",
read "H_1 *is supported by* E_1 *to the same extent that* H_2 *is supported by* E_2";

(3) $H_1|E_1 \asymp H_2|E_2$ abbreviates "not $H_1|E_1 \succcurlyeq H_2|E_2$ and not $H_2|E_2 \succcurlyeq H_1|E_1$",
read "*the support for* H_1 *by* E_1 *is not determinately comparable to that of* H_2 *by* E_2";

(4) $E \Rightarrow H$ abbreviates "$H|E \succcurlyeq E|E$";
read "E *supportively entails* H".

It will turn out that the axioms for a relation $\succcurlyeq$ make the corresponding *supportive entailment relation* $\Rightarrow$ satisfy the rules for a well-known kind of non-monotonic conditional called a *rational consequence relation*. Indeed, each of the *rational consequence relations* turns out to be represented by some *supportive entailment relation* (for some relation $\succcurlyeq$ that satisfies the axioms below). Furthermore, *classical logical entailment* will turn out to be extensionally equivalent to *supportive entailment for every relation* $\succcurlyeq$.[7]

With these definitions in place we are now ready to specify the axioms for the *comparative support relations*. We begin with a class of relations that is a bit too broad – relations that satisfy some probability-like rules, but which are not sufficiently constrained so as to always behave like probabilistic comparisons.

Let *L* be a language having the syntax of sentential logic. Each *proto-support relation* $\succcurlyeq$ is a binary relation between pairs of sentences that satisfies the following axioms:

0. for some H_1, E_1, H_2, E_2, $H_1|E_1 \succ H_2|E_2$ (non-triviality)
[Not all arguments are equally strong; at least one is stronger than at least one other. This corresponds to axiom 0 for the *Popper Functions*.]

1. $E|E \succcurlyeq H|E$ (maximality)
[Self-support is maximal support; it's at least as strong as any other argument strength, and all arguments are *comparable in strength* to such a maximal argument. This corresponds to axiom 1 for the *Popper Functions*, but is a bit stronger. The fact that the *Popper functions* are

[7] For each *Popper function* P, define the corresponding *Popper-entailment relation* $\rightarrow$ such that for each pair of sentences A and B, '$B \rightarrow A$' holds just when $P[A \mid B] = 1$. Then, similarly, it turns out that: (1) each *Popper-entailment relation* is a *rational consequence relation*; (2) every *rational consequence relation* is a *Popper-entailment relation* for some *Popper function* P; and (3) *classical logical entailment* is extensionally equivalent to *Popper-entailment for every Popper function* P. Thus, the ranked structure of the *Popper functions* (structured as a hierarchy of classical probability functions) is just the ranked structure of the *rational consequence relations*. The *comparative support relations* turn out to share this ranked structure, which is captured by their associated *supportive entailment relations*.

ordered by the real numbers automatically implies that all conditional probabilities are comparable to self-support probabilities.]

2. If $H_1|E_1 \succcurlyeq H_2|E_2$ and $H_2|E_2 \succcurlyeq H_3|E_3$, then $H_1|E_1 \succcurlyeq H_3|E_3$ (transitivity)
   [Comparative argument strength is transitive. This the *Popper Functions* get this property from the ordering of the real numbers.]

3. $H|(E_1 \cdot E_2) \succcurlyeq H|(E_2 \cdot E_1)$ (antecedent commutivity)
   [We don't *assume* that logically equivalent statements support other statements equally well; but that will turn out to be provable from this together with the others. This corresponds to axiom 3 for the *Popper Functions*.]

4.1 $(H \cdot H)|E \succcurlyeq H|E$ (consequent repetition)

4.2 $H_1|E \succcurlyeq (H_1 \cdot H_2)|E$ (simplification)
   [We don't *assume* that whenever B *logically entails* A, *the support for* A *by* E *is at least as strong as the support for* B *by* E; but that will be provable from these two axioms together with the others. 4.2 corresponds to axiom 2 for the *Popper Functions*.]

5.1 If $H_1|E_1 \succcurlyeq {\sim}H_2|E_2$, then $H_2|E_2 \succcurlyeq {\sim}H_1|E_1$ unless $E_1 \Rightarrow D$ for all D

5.2 If ${\sim}H_1|E_1 \succcurlyeq H_2|E_2$, then ${\sim}H_2|E_2 \succcurlyeq H_1|E_1$ unless $E_1 \Rightarrow D$ for all D (negation-symmetry)
   [Since this logic doesn't presuppose classical deductive logic, these two axioms effectively define the inferential meaning of the '~' symbol. The first says that whenever H_1 *is supported by* E_1 *at least as strongly as the falsity of* H_2 *is supported by* E_2, then H_2 *is supported by* E_2 *at least as strongly as the falsity of* H_1 *is supported by* E_1; the only exception is in cases where premise E_1 behaves like a contradiction and maximally supports every statement D. The second is similar. These axioms turn out to capture the essence of the additivity axiom for conditional probability, axiom 5 for the *Popper functions*.]

6.1 If $H_1|(A_1 \cdot E_1) \succcurlyeq H_2|(A_2 \cdot E_2)$ and $A_1|E_1 \succcurlyeq A_2|E_2$, then $(H_1 \cdot A_1)|E_1 \succcurlyeq (H_2 \cdot A_2)|E_2$

6.2 If $H_1|(A_1 \cdot E_1) \succcurlyeq A_2|E_2$ and $A_1|E_1 \succcurlyeq H_2|(A_2 \cdot E_2)$, then $(H_1 \cdot A_1)|E_1 \succcurlyeq (H_2 \cdot A_2)|E_2$ (composition)
   [These two axioms together with the next four capture the essential content of probabilistic conditionalization, which is expressed by axiom 4 for the *Popper functions*. Axioms 6 represent *comparative support strength* versions of a principle of *nonmonotionic support strength* called *cumulative transitivity*, which is a basic principle of many nonmonotonic logics: if $E \rightarrow A$ and $(E \cdot A) \rightarrow H$, then $E \rightarrow (A \cdot H)$.]

7.1 If $(H_1 \cdot A_1)|E_1 \succcurlyeq (H_2 \cdot A_2)|E_2$ and $A_2|E_2 \succcurlyeq A_1|E_1$ and $E_2 \not\Rightarrow {\sim}A_2$, then $H_1|(A_1 \cdot E_1) \succcurlyeq H_2|(A_2 \cdot E_2)$

7.2 If $(H_1 \cdot A_1)|E_1 \succcurlyeq (H_2 \cdot A_2)|E_2$, then $A_2|E_2 \succcurlyeq H_1|(A_1 \cdot E_1)$ and $E_2 \not\Rightarrow {\sim}A_2$, then $A_1|E_1 \succcurlyeq H_2|(A_2 \cdot E_2)$

7.3 If $(H_1 \cdot A_1)|E_1 \succcurlyeq (H_2 \cdot A_2)|E_2$, then $H_2|(A_2 \cdot E_2) \succcurlyeq H_1|(A_1 \cdot E_1)$ and $(A_2 \cdot E_2) \not\Rightarrow {\sim}H_2$, then
   $A_1|E_1 \succcurlyeq A_2|E_2$

7.4 If $(H_1 \cdot A_1)|E_1 \succcurlyeq (H_2 \cdot A_2)|E_2$, then $H_2|(A_2 \cdot E_2) \succcurlyeq A_1|E_1$ and $(A_2 \cdot E_2) \not\Rightarrow {\sim}H_2$, then
$H_1|(A_1 \cdot E_1) \succcurlyeq A_2|E_2$ (decomposition)

[These four axioms capture the idea that the support strength for a conjunction $(H \cdot A)$ by evidence claim E is essentially dependent on how strongly H is supported by the conjunction $(A \cdot E)$ *and* the strength with which A is itself supported by E. Thus, when comparing the support strengths of two conjunctions by respective evidence claims, $(H_1 \cdot A_1)|E_1$ and $(H_2 \cdot A_2)|E_2$, if the first argument is *as strong or stronger than* the second argument, then it should not be possible that both component parts, $H_1|(A_1 \cdot E_1)$ and $A_1|E_1$, of the first argument are weaker than both component parts, $H_2|(A_2 \cdot E_2)$ and $A_2|E_2$, of the possibly weaker argument. The only possible exceptions are cases where one of the components of the weaker conjunctive argument is absolutely minimal – i.e. where either $E_2 \Rightarrow {\sim}A_2$ or $(A_2 \cdot E_2) \Rightarrow {\sim}H_2$. For example, consider axiom 7.1 when $A_2|E_2 \succcurlyeq A_1|E_1$ but also $E_2 \Rightarrow {\sim}A_2$. We can derive that $E_1 \Rightarrow {\sim}A_1$ as well. It then follows that both of the conjunctive arguments are absolutely minimal in strength as well: $E_1 \Rightarrow {\sim}(H_1 \cdot A_1)$ and $E_2 \Rightarrow {\sim}(H_2 \cdot A_2)$. In such a case there is not requirement on the comparative strengths of the remaining two components, $H_1|(A_1 \cdot E_1)$ and $H_2|(A_2 \cdot E_2)$, of the two conjunctive arguments. By analogy, when $P[H_1 \cdot A_1 \mid E_1] \geq P[H_2 \cdot A_2 \mid E_2]$ and it's also the case that $0 = P[A_2 \mid E_2] \geq P[A_1 \mid E_1]$, we must have $0 = P[H_1 \cdot A_1 \mid E_1] = P[H_2 \cdot A_2 \mid E_2]$, in which case the *Popper functions* impose no constrains on the value of $P[H_1 \mid A_1 \cdot E_1]$ and $P[H_2 \mid A_2 \cdot E_2]$. A similar analysis applies to the "exceptional clauses", $E_2 \not\Rightarrow {\sim}A_2$ and $(A_2 \cdot E_2) \not\Rightarrow {\sim}H_2$, of 7.2-7.4.]

8. If $H|(E \cdot F) \succcurlyeq A|B$ and $H|(E \cdot {\sim}F) \succcurlyeq A|B$, then $H|E \succcurlyeq A|B$ (alternate presumption)*
[Probabilistically this axiom follows from additivity together with conditionalization – i.e. since $P[H \mid E] = P[H \mid E \cdot F] \times P[F \mid E] + P[H \mid E \cdot {\sim}F] \times (1 - P[F \mid E])$, if both $P[H \mid E \cdot F] \geq r$ and $P[H \mid E \cdot {\sim}F] \geq r$, then $P[H \mid E] \geq r$. Axiom 8 is a qualitative version of this result. This axiom can be proved from the other axioms if the relation $\succcurlyeq$ is assumed to be *completely comparable* rather than merely a *partial order relation*; the asterisks, '*', on the name of this axiom is there as a reminder of its derivability in the presence of *completeness*.]

All relations $\succcurlyeq$ that satisfy these axioms are *weak partial orders* – i.e. they are transitive and reflexive. Transitivity is guaranteed by axiom 2; reflexivity, $H|E \succcurlyeq H|E$, follows easily from axioms 4.1, 4.2, and 2. I call the relations that satisfy these axioms *proto-support relations* because the axioms still need a bit of strengthening to rule out some relations $\succcurlyeq$ that fail to fully behave as *comparative support relations* should. I'll say more about that later.

The asterisks on the name of axiom 8 is there to indicate that it follows from the other axioms whenever the relation $\succcurlyeq_\alpha$ is also *complete* – i.e. whenever, for all pairs of sentence pairs, the following *complete comparability rule* also holds (or is added as an additional axiom) for relation $\succcurlyeq$:

either $H_1|E_1 \succcurlyeq H_2|E_2$ or $H_2|E_2 \succcurlyeq H_1|E_1$ (complete comparability).

This rule would require that all argument pairs are comparable in strength: $H_1|E_1 \not\asymp H_2|E_2$. Any relation $\succcurlyeq$ that satisfies the above axioms together with *complete comparability* is a *weak order relation* rather than merely a *weak partial order relation*.[8]

Notice that none of these axioms draws on the notion of *logical entailment*. Rather, the logic of the *proto-support relations* is autonomous from classical logic, just as the axiomatization of the *Popper functions* provided earlier is autonomous. A notion of *logical entailment* falls out of the logic of the *proto-support relations* as a special case, and it then turns out to be extensionally equivalent to the classical notion of *logical entailment*:

Definition: B *logically-supportively-entails* A (abbreviated 'B |=> A') just in case for all *proto-support relations* $\succcurlyeq$, $B \Rightarrow A$ (i.e. $A|B \succcurlyeq B|B$).

Theorem: B |=> A if and only if B |= A (i.e. iff B *classically logically entails* A).

This theorem takes some work to prove, but once obtained, the following rules also result:

{3}. If $E_1 \models E_2$ and $E_2 \models E_1$, then $H|E_2 \succcurlyeq H|E_1$ (classical antecedent equivalence)
{4}. If $(E \cdot H_2) \models H_1$, then $H_1|E \succcurlyeq H_2|E$ (classical consequent entailment)
{5}. $H_1|E_1 \succcurlyeq H_2|E_2$, then $\sim H_2|E_2 \succcurlyeq \sim H_1|E_1$ unless $E_1 \Rightarrow D$ for all D (weak negation-symmetry)

Furthermore, the system of axioms gotten by replacing axioms 3, 4.1, 4.2, 5.1, and 5.2 with these three rules can be used to derive axioms 3-5.2, and so provides an equivalent alternative axiomatization of the *proto-support relations*.[9]

[8] Koopman also provides the following rule as an axiom:
For any integer $n > 1$, if $A_1, ..., A_n$, and $B_1, ..., B_n$ are collections of sentences such that
$C \not\Rightarrow \sim C$, $C \Rightarrow (A_1 \vee ... \vee A_n)$, $C \Rightarrow \sim(A_i \cdot A_j)$, $A_n|C \succcurlyeq ... \succcurlyeq A_2|C \succcurlyeq A_1|C$, and
$D \not\Rightarrow \sim D$, $D \Rightarrow (B_1 \vee ... \vee B_n)$, $D \Rightarrow \sim(B_i \cdot B_j)$, $B_n|D \succcurlyeq ... \succcurlyeq B_2|D \succcurlyeq B_1|D$,
then $A_n|C \succcurlyeq B_1|D$ (subdivision)*
This rule may not seem as intuitively compelling as the others, so I forego it here. Later we will want to require that comparative support relations be *extendable* it *completely comparable* relations. The rule of *subdivision* is derivable from axioms 1-7.4 in the presence of *complete comparability*.

[9] To extend the proto-support relations to predicate logic (with identity), we may do something analogous to the treatment that extend to *Popper functions* to the *Popper-Field functions*.
Let L_0 be a language with the syntax for predicate logic (with identity), and consider a class of languages $\boldsymbol{L}$ that contains L_0 together with name-extensions of L_0 drawn from a countable set of new names. Define a *PS-class* for *proto-support relations* to be a set *M* of relations such that each of them, $\succcurlyeq_\alpha$, satisfy axioms 0-9 together with the following axioms:
For all sentences C, formulas Fx with only x free, and names $c_1, ..., c_m$ in $\succcurlyeq_\alpha$'s language:
(i) $((Fc_1 \cdot Fc_2) \cdot ... \cdot Fc_m)| B \succcurlyeq_\alpha \forall xFx|B$
(ii) if $A|C \succ_\alpha \forall xFx|B$, then for some $\succcurlyeq_\beta$ in *M* on a name extension of $\succcurlyeq_\alpha$'s language (that agrees with $\succcurlyeq_\alpha$ on its language), for some names $e_1, ..., e_n$ in $\succcurlyeq_\beta$'s language, $A|C \succcurlyeq_\beta ((Fe_1 \cdot Fe_2) \cdot ... \cdot Fe_n)|B$

Notice that each of these axioms is *probabilistically sound* in the following sense:

> For each *Popper function* P, define the *corresponding comparative relation* $\succcurlyeq$ to be the relation such that, for all sentences H_1, E_1, H_2, E_2, $H_1|E_1 \succcurlyeq H_2|E_2$ if and only if $P[H_1 \mid E_1] \geq P[H_2 \mid E_2]$.
> Then for each *Popper function*, the *corresponding comparative relation* can be shown to be a *proto-support relation* – i.e. it satisfies the above axioms.

The axioms that constrain the "full" *comparative support relations* (which is a restricted subset of the *proto-support relations*) will turn out to be *probabilistically complete* in the sense that each full *comparative support relation* $\succcurlyeq$ (that satisfies the above axioms plus a bit more) will be *representable* by a *Popper function* P whose *corresponding comparative relation* agrees with $\succcurlyeq$ (or at least "agrees very nearly"). However, the axioms for the *proto-support relations* are not *probabilistically complete*. It turns out that some *proto-support relations* are not be probability-like enough to be representable by any of the *Popper functions*.

The *proto-support relations* are sufficiently strong to provide comparative forms of Bayes' theorem.

> Bayes' Theorem 1: Suppose $B \not\Rightarrow {\sim}H_1$.
> If $E|(H_1{\cdot}B) \succ E|(H_2{\cdot}B)$ and $H_1|B \succcurlyeq H_2|B$, then $H_1|(E{\cdot}B) \succ H_2|(E{\cdot}B)$.

This is a comparative analogue of a ratio version of Bayes' theorem. It expresses the influence of ratios of likelihoods, $P[E|\,(H_2{\cdot}B)] / P[E|\,(H_1{\cdot}B)]$, together with ratios of prior probabilities, $P[H_2 \mid B] / P[H_1 \mid B]$, on the comparative values of posterior probabilities, $P[H_2 \mid (E{\cdot}B)] / P[H_1 \mid (E{\cdot}B)]$. The condition '$B \not\Rightarrow {\sim}H_1$' corresponds to '$P[H_1 \mid B] > 0$'; and '$E|(H_1{\cdot}B) \succ E|(H_2{\cdot}B)$' implies a condition analogous to '$P[E|\,(H_1{\cdot}B)] > P[E|\,(H_2{\cdot}B)] \geq 0$'. From a relevant probabilistic theorem,

$$P[H_2 \mid (E{\cdot}B)] / P[H_1 \mid (E{\cdot}B)] \;=\; (P[E|\,(H_2{\cdot}B)] / P[E|\,(H_1{\cdot}B)]) \;\times\; (P[H_2 \mid B] / P[H_1 \mid B]),$$

we may derive the analogous probabilistic result:

$$\text{if } P[E|\,(H_1{\cdot}B)] > P[E|\,(H_2{\cdot}B)] \text{ and } P[H_1 \mid B] \geq P[H_2 \mid B], \text{ then } P[H_1 \mid (E{\cdot}B)] > P[H_2 \mid (E{\cdot}B)].$$

(iii) if for every $\succcurlyeq_\beta$ in *M* a name extension of $\succcurlyeq_\alpha$ that agrees with $\succcurlyeq_\alpha$ on its language, $((Fe_1{\cdot}Fe_2){\cdot}\ldots{\cdot}Fe_n)|B \succ_\beta A|C$ for all n and names $e_1, \ldots, e_n$ in $\succcurlyeq_\beta$'s language, then $\forall xFx \succcurlyeq_\alpha A|C$. (When $\succcurlyeq_\alpha$ satisfies the *complete comparability rule*, (iii) is derivable from (ii).)
Now define the class ***PS*** of all *proto-support functions* on languages in ***L*** to be the union of all such *PS-classes M*. (Axiom (iii) follows from axiom (ii) if the relation $\succcurlyeq_\alpha$ is *complete*.)
To include identity, require members of ***PS*** to also satisfy the following two axioms:
(iv) $\forall x\; x{=}x\, |B \;\succcurlyeq_\alpha\; B|B$
(v) $Fb \mid ((Fa \cdot b{=}a) \cdot B) \;\succcurlyeq_\alpha\; B|B$.
Given these axioms, the above definition of *logically-supportive-entailment* and the theorem that it coincides with *classical logical entailment* applies to predicate logic (with identity). Then {3}-{5} and all of the following results apply to predicate logic (with identity).

Here is a second Bayesian result that derives from the logic of proto-support relations.

Bayes' Theorem 2: Suppose $B \not\Rightarrow \sim H_1$ and $B \not\Rightarrow \sim H_2$ and $B \Rightarrow \sim(H_1 \cdot H_2)$.
If $E|(H_1 \cdot B) \succ E|(H_2 \cdot B)$, then
$H_1|(E \cdot B \cdot (H_1 \vee H_2)) \succ H_1|(B \cdot (H_1 \vee H_2))$ and $H_2|(B \cdot (H_1 \vee H_2)) \succ H_2|(E \cdot B \cdot (H_1 \vee H_2))$.

This theorem has a straightforward probabilistic analogue. It says that when $P[H_1 \mid B] > 0$, $P[H_2 \mid B] > 0$, and $P[H_1 \cdot H_2 \mid B] = 0$, if the likelihoods $P[E|(H_1 \cdot B)] > P[E|(H_2 \cdot B)]$, then we must have the following posterior probabilities when comparing H_1 directly to H_2:
$P[H_1 \mid (E \cdot B) \cdot (H_1 \vee H_2)] > P[H_1 \mid B \cdot (H_1 \vee H_2)]$ and $P[H_2 \mid (E \cdot B) \cdot (H_1 \vee H_2)] < P[H_2 \mid B \cdot (H_1 \vee H_2)]$.
This is a comparative probabilistic way of expressing the relationship,
$P[H_2 \mid (E \cdot B)] / P[H_1 \mid (E \cdot B)] < P[H_2 \mid B] / P[H_1 \mid B]$.

Now let's see what's needed to restrict the class of *proto-support relations* to a class of *comparative support relations*, where each such *comparative support relation* behaves so much like a comparison between conditional probabilities that it is *representable* by a *Popper function* that provides the same comparative structure between argument pairs.

4. The *Comparative Evidential Support Relations* and their Probabilistic Representations

Consider the following additional rules. We will employ these rules as axioms to characterize classes of *comparative support relations* that behave like comparisons between conditional probability values.

9. $H_1|E_1 \succcurlyeq H_2|E_2$ or $H_2|E_2 \succcurlyeq H_1|E_1$ (complete comparability)
   [A relation $\succcurlyeq$ that satisfies this rule is a weak order rather than merely a partial order; such a relation $\succcurlyeq$ compares the strength of each possible argument to each of the others.]

10. For each integer $m \geq 2$ there is an integer $n \geq m$ such that for n sentences $S_1, \ldots, S_n$ and some sentence G: $G \not\Rightarrow \sim S_1$, and for all distinct i, j, $G \Rightarrow \sim(S_i \cdot S_j)$ and $S_i|G \approx S_j|G$.
    (existence of arbitrarily large equal-partitions)
    [When such sentences G and n-tuples of sentences S_i exist for a relation $\succcurlyeq$, it can be shown that: $(G \cdot (S_1 \vee \ldots \vee S_n)) \not\Rightarrow \sim S_1$, for i, j, $(G \cdot (S_1 \vee \ldots \vee S_n)) \Rightarrow \sim(S_i \cdot S_j)$, $S_i|(G \cdot (S_1 \vee \ldots \vee S_n)) \approx S_j|(G \cdot (S_1 \vee \ldots \vee S_n))$, and also $(G \cdot (S_1 \vee \ldots \vee S_n)) \Rightarrow (S_1 \vee \ldots \vee S_n)$. Thus, for each n, $\succcurlyeq$ has an exclusive and exhaustive equal-partition based on $(G \cdot (S_1 \vee \ldots \vee S_n))$. These partitions can be used to provide approximate "probabilistic values" for the strengths of arguments H|E:
    when $(S_1 \vee \ldots \vee S_{m+1})|(G \cdot (S_1 \vee \ldots \vee S_n)) \succcurlyeq H|E \succcurlyeq (S_1 \vee \ldots \vee S_m)|(G \cdot (S_1 \vee \ldots \vee S_n))$
    we effectively get a probabilistic approximation: $(m+1)/n \geq P[H \mid E] \geq m/n$.
    For arbitrarily large partitions (for arbitrarily large values of n) we get arbitrarily close probabilistic bounds on the strength of each argument.]

10^{+}. If $H_1|E_1 \succ H_2|E_2$, then for some $n \geq 2$ there are n sentences $S_1, ..., S_n$ and a sentence F such that:
$F \not\Rightarrow \sim S_1$, and for distinct i, j, $F \Rightarrow \sim(S_i \cdot S_j)$ and $S_i|F \approx S_j|F$, and $F \Rightarrow (S_1 \vee ... \vee S_n)$, and for some m of them, $H_1|E_1 \succ (S_1 \vee \ldots \vee S_m)|F \succ H_2|E_2$. (Archimedean equal-partitions)

[Rule 10^{+} can be shown to imply rule 10, given the other axioms. In addition, 10^{+} supplies a kind of "Archimedean condition": it requires that whenever $H_1|E_1 \succ H_2|E_2$, there is an equal-partition that, for sufficiently large n, squeezes a "strength comparison" between $H_1|E_1$ and $H_2|E_2$, which forces these two arguments to exhibit distinct probabilistically strength values:

$P[H_1 \mid E_1] > m/n > P[H_2 \mid E_2]$.

Those comparative support relations that satisfy rules 9 and 10 but fail to satisfy 10^{+} must permit some argument pairs such that $H_1|E_1 \succ H_2|E_2$ to be infinitesimally close together in comparative strength – no equal-partition argument can be fitted between $H_1|E_1$ and $H_2|E_2$.]

Rule 10^{+} implies rule 10 (given the other axioms). Rule 10 together with the other axioms is equivalent to the following slightly stronger claim (which could be used in place of rule 10):

10*. For each integer $m \geq 2$ there is an integer $n \geq m$ such that for n sentences $S_1, ..., S_n$ and some sentence F: $F \not\Rightarrow \sim S_1$, for distinct i, j, $F \Rightarrow \sim(S_i \cdot S_j)$, $F \Rightarrow (S_1 \vee ... \vee S_n)$ and $S_i|F \approx S_j|F$.

Here is an intuitive example of statements that satisfy rule 10*. Let statement F describe a completely fair lottery consisting of exactly n tickets. Each of the sentences S_i says "ticket i will win". F *says* (*supportively entails*, and may even *logically entail*):

(1) "at least one ticket will win": so $F \Rightarrow (S_1 \vee \ldots \vee S_n)$;
(2) "no two tickets will win": so $F \Rightarrow \sim(S_i \cdot S_j)$, for each distinct pair of claims S_i and S_j;
(3) "each ticket has the same chance of winning" (so, the argument from F provides exactly the same support for the claim "ticket i will win" as for the claim "ticket j will win"): $S_i|F \approx S_j|F$ for each distinct pair of claims S_i and S_j. (Furthermore, we suppose that F does not *supportively entail* "ticket 1 won't win", i.e. $F \not\Rightarrow \sim S_1$, which formally reduces to $F|F \succ \sim S_1|F$; this eliminates the possibility that F behaves like a contradiction, saying both that "none of the tickets wins" and that "one of the tickets will win").

We might have just required all *comparative support relations* of interest to satisfy rule 10 (or 10*). This would be an intuitively rather plausible requirement – it would merely require that the language of each relation $\succcurlyeq$ have the ability to describe various such lotteries for arbitrarily large finite numbers of tickets. (Presumable our own language can do that.) There is no requirement that such lotteries actually exist – rule 10 (and 10*) is only about the comparative *support strength* implications of such lottery descriptions G (and F), not about the truth of these claims. Nevertheless, we won't require all *comparative support relations* to contain such lottery descriptions. Rather, it will suffice for our purposes that all such relations merely be *extendable* to relations that contain such lottery descriptions (or similar rule 10 satisfying statements). I'll say more about that momentarily.

Now, consider all of the *proto-support relations* – i.e. all those relations that satisfy axioms 0-8. Some of these relations also (already) satisfy one or more of the additional rules, 9, and 10 or 10^+. I'll assign names to special classes of these relations. But before doing that I need to introduce one additional idea.

It is distinctly desirable that a logic of *comparative evidential support* not require every argument to be *determinately comparable* in strength to every other argument. For instance, where H_1 and H_2 are alternative cosmological theories, there may be no fact of the matter as to whether a tautology T supports one of them more or less strongly than the other, so we want it to be possible for incomparability to hold between them, $H_1|T \asymp_\alpha H_2|T$, contra rule 9. Rather, only certain additional considerations (both evidential considers, E, and broadly empirical arguments and background knowledge, B) will yield meaningful comparisons of argumentative support strength (e.g. perhaps $H_1|B \succ_\alpha H_2|B$, and then, in some cases where $E|(H_2{\cdot}B) \succ_\alpha E|(H_1{\cdot}B)$, we may have $H_2|(E{\cdot}B) \succ_\alpha H_1|(E{\cdot}B)$). Indeed, there are lots of cases where distinct arguments are simply incomparable in strength, where neither is distinctly stronger than the other, but where they are not definitely equal in strength either. Thus, in many cases a *comparative support relation* should quite legitimately violate the *complete comparability* rule. However, I will argue that each legitimate comparative support relation should in principle be *extendable* to a *complete* relation. I'll provide the argument for that in a moment. Let's first define the relevant notion of *extendability*.

Definition: A *proto-support relation* $\succcurlyeq_\alpha$ is *extendable* to a *proto-support relation* $\succcurlyeq_\beta$ just when the language of $\succcurlyeq_\beta$ contains the language of $\succcurlyeq_\alpha$ (i.e. contains the same syntactic expressions, and perhaps additional expressions as well) and the following two conditions hold:
(1) whenever $A|B \succ_\alpha C|D$, then also $A|B \succ_\beta C|D$;
(2) whenever $A|B \approx_\alpha C|D$, then also $A|B \approx_\beta C|D$.

The idea is that when $\succcurlyeq_\alpha$ is *extendable* to $\succcurlyeq_\beta$, all argument pairs that are *comparable* according to $\succcurlyeq_\alpha$ must also *compare in the same way* according to $\succcurlyeq_\beta$. However, the language on which $\succcurlyeq_\beta$ expresses and compares arguments may be broader than the language for $\succcurlyeq_\alpha$, and $\succcurlyeq_\beta$ may *compare* argument pairs not considered to be *comparable* according $\succcurlyeq_\alpha$.[10] In a moment I will argue that although it is implausibly strong to require *comparative support relations* to be *complete* (to *compare* all argument pairs), it is quite reasonable to require each *comparative support relation* to be *extendable* to a *complete* relation. But before discussing that, let's now

[10] This definition counts each relation $\succcurlyeq_\alpha$ as a trivial *extension* of itself. It also permits $\succcurlyeq_\beta$ to be defined on precisely the same language as $\succcurlyeq_\alpha$, and to merely *extend* $\succcurlyeq_\alpha$ by *comparing* some arguments that are *incomparable* according to $\succcurlyeq_\alpha$. More generally, $\succcurlyeq_\beta$ may also include comparisons among sentences not expressible in the syntax of the language of $\succcurlyeq_\alpha$. Also notice that the relationship between $\succcurlyeq_\alpha$ and an *extension* of it $\succcurlyeq_\beta$ is purely syntactic. There is no presumption that a relation $\succcurlyeq_\alpha$ and an *extension* of it $\succcurlyeq_\beta$ have to be associated with the same meaning assignments to the sentences they share.

assign names to those classes of comparative support relations that will play an important role throughout the remainder of the discussion.

Definitions of Classes of *Comparative Support Relations*:

1. A ***complete-Archimedean*** *comparative support relation* is any *proto-support relation* that satisfies rules 9 and 10^+.
2. A ***complete*** *comparative support relation* is any *proto-support relation* that satisfies rules 9 and 10.
3. A ***potentially-Archimedean*** *comparative support relation* is any *proto-support relation* that is *extendable* to a *proto-support relation* that satisfies rules 9 and 10^+ (i.e. that is *extendable* to a ***complete-Archimedean*** *comparative support relation*).
4. A *comparative support relation* (*simpliciter*) is any *proto-support relation* that is *extendable* to a *proto-support relation* that satisfies rules 9 and 10 (i.e. that is *extendable* to a ***complete*** *comparative support relation*).

Given that rule 10^+ implies rule 10 (and that each relation counts as a trivial *extension* of itself), all of the more specific kinds of *comparative support relations* specified by clauses 1-3 count as *comparative support relations* as specified by clause 4. Furthermore, each ***complete-Archimedean*** *relation* is a ***complete*** *relation* (as specified by 2), and also counts as a ***potentially-Archimedean*** relation (as specified by 3). However, some ***complete*** *relations* are not ***potentially-Archimedean*** *relations* (we'll see an example later)**.** Thus, each relation of the kind specified by a lower-numbered clause is also among the class of relations specified by a higher-numbered clause, but with the single exception that some ***complete*** *relations* are not ***potentially-Archimedean*** *relations***.** Furthermore, each class of relations specified by higher-numbered clauses contains some relations that don't satisfy the more restrictive conditions of the lower-numbered clauses.

As I've already suggested, the requirement that a *comparative support relation* be *complete* is much too strong. However, the *complete* relations are interesting because they act exactly like comparisons of conditional probabilities. Indeed, each *complete* relation corresponds to (is modeled by) a *unique Popper function*, whereas an *incomplete* relation that merely *extendable* to *complete* relation will often correspond to (i.e. be modeled by) a large class of *Popper functions*, where each *Popper function* in the class is an equally good model of the ordering among argument pairs specified by the *comparative support relation* it represents. Here is a precise statement of the way in which *Popper functions* can *represent* the various kinds of *comparative support relations* specified in the previous definition.

Representation Theorems for Classes of *comparative support relations*:

1. For each ***complete-Archimedean*** *comparative support relation* $\succcurlyeq$ there is a unique *Popper function* P such that for all H_1, E_1, H_2, E_2 in the language for $\succcurlyeq$,

   $P[H_1 \mid E_1] \geq P[H_2 \mid E_2]$ if and only if $H_1|E_1 \succcurlyeq H_2|E_2$.

   Note 1: given the *completeness* of $\succcurlyeq$, this condition is equivalent to the conjunction of the following conditions:

   (1) if $P[H_1 \mid E_1] > P[H_2 \mid E_2]$, then $H_1|E_1 \succ H_2|E_2$;

   (2) if $P[H_1 \mid E_1] = P[H_2 \mid E_2]$, then $H_1|E_1 \approx H_2|E_2$.

Note 2: given the *completeness* of $\succcurlyeq$, this condition is also equivalent to the conjunction of the following conditions:

(1) if $H_1|E_1 \succ H_2|E_2$, then $P[H_1 \mid E_1] > P[H_2 \mid E_2]$;

(2) if $H_1|E_1 \approx H_2|E_2$, then $P[H_1 \mid E_1] = P[H_2 \mid E_2]$.

2. For each ***complete*** *comparative support relation* $\succcurlyeq$ there is a unique *Popper function* P such that for all H_1, E_1, H_2, E_2 in the language for $\succcurlyeq$,

   if $P[H_1 \mid E_1] > P[H_2 \mid E_2]$, then $H_1|E_1 \succ H_2|E_2$.

   Note: given the *completeness* of $\succcurlyeq$, this condition is equivalent to the conjunction of the following conditions:

   (1) if $H_1|E_1 \succ H_2|E_2$, then $P[H_1 \mid E_1] \geq P[H_2 \mid E_2]$;

   (2) if $H_1|E_1 \approx H_2|E_2$, then $P[H_1 \mid E_1] = P[H_2 \mid E_2]$.

3. For each ***potentially-Archimedean*** *comparative support relation* $\succcurlyeq$ there is a (not necessarily unique) *Popper function* P such that for all H_1, E_1, H_2, E_2 in the language for $\succcurlyeq$,

   if not $H_1|E_1 \asymp H_2|E_2$, then $P[H_1 \mid E_1] \geq P[H_2 \mid E_2]$ if and only if $H_1|E_1 \succcurlyeq H_2|E_2$.

   Note 1: this condition is equivalent to the conjunction of the following conditions:

   (1) if $P[H_1 \mid E_1] > P[H_2 \mid E_2]$, then $H_1|E_1 \succ H_2|E_2$ or $H_1|E_1 \asymp H_2|E_2$;

   (2) if $P[H_1 \mid E_1] = P[H_2 \mid E_2]$, then $H_1|E_1 \approx H_2|E_2$ or $H_1|E_1 \asymp H_2|E_2$.

   Note 2: this condition is also equivalent to the conjunction of the following conditions:

   (1) if $H_1|E_1 \succ H_2|E_2$, then $P[H_1 \mid E_1] > P[H_2 \mid E_2]$;

   (2) if $H_1|E_1 \approx H_2|E_2$, then $P[H_1 \mid E_1] = P[H_2 \mid E_2]$.

4. For each *comparative support relation* $\succcurlyeq$ there is a (not necessarily unique) *Popper function* P such that for all H_1, E_1, H_2, E_2 in the language for $\succcurlyeq$,

   if $P[H_1 \mid E_1] > P[H_2 \mid E_2]$, then $H_1|E_1 \succ H_2|E_2$ or $H_1|E_1 \asymp H_2|E_2$.

   Note: this condition is equivalent to the following pair of conditions:

   (1) if $H_1|E_1 \succ H_2|E_2$, then $P[H_1 \mid E_1] \geq P[H_2 \mid E_2]$;

   (2) if $H_1|E_1 \approx H_2|E_2$, then $P[H_1 \mid E_1] = P[H_2 \mid E_2]$.

The ***complete*** *comparative support relations* (including the ***complete-Archimedean*** *relations*) cannot accommodate *incomparability* in cases where comparative argument strength really should be incomparable. In this respect they are exactly like probability functions. Indeed, the representation theorem shows that each ***complete-Archimedean*** relation is virtually identical to its uniquely representing *Popper function*.[11] Furthermore, those ***complete*** relations that are not ***complete-Archimedean*** only fall short of this kind of virtual identity to a probability function in those cases where one argument is merely *infinitesimally stronger* than another. Such argument

[11] Whenever the representing *Popper function* P for ***complete-Archimedean*** *relation* $\succcurlyeq$ assigns $P[H \mid E] = r$ for a rational number $r = m/n$, relation $\succcurlyeq$ itself acts precisely like the representing probability function via a rule 10 satisfying partition for which $H|E \approx (S_1\vee\ldots\vee S_m)|(G\cdot(S_1\vee\ldots\vee S_n))$ [i.e. for which $(S_1\vee\ldots\vee S_m)|(G\cdot(S_1\vee\ldots\vee S_n)) \succcurlyeq H|E \succcurlyeq (S_1\vee\ldots\vee S_m)|(G\cdot(S_1\vee\ldots\vee S_n))$]. When the representing function P for $\succcurlyeq$ assigns $P[H \mid E] = r$ for an irrational number r, then the relation $\succcurlyeq$ would has it that a sequence of increasingly large rule 10 satisfying partitions (for ever larger n) yields relationships $(S_1\vee\ldots\vee S_{m+1})|(G\cdot(S_1\vee\ldots\vee S_n)) \succcurlyeq H|E \succcurlyeq (S_1\vee\ldots\vee S_m)|(G\cdot(S_1\vee\ldots\vee S_n))$, where the values of m/n and (m+1)/n converge to r (for appropriate values of m) as n increases.

pairs are so close in comparative strength that the unique representing *Popper function* has to assign them identical numerical values. (One might be tempted to just toss out relations of this sort, and require all of the *complete* relations of interest to be ***complete-Archimedean*** *relations*; but it turns out that some of the non-Archimedean relations capture intuitively plausible cases where some arguments should indeed be only infinitesimally stronger than some others. I'll describe an example in a moment.) So, although the ***complete*** *relations* have all of the benefits of conditional probability functions, as representations of argument strength they also have the same drawbacks; they must *compare* argument pairs that really should be *incomparable* in strength.

The *proto-support relations* commonly take a wide array of argument pairs to be *incomparable in strength*. Among them, I've only deemed those relations that can at least in principle be *extended* to *complete relations* (which provide definite strength comparisons between all argument pairs) to count as full-fledged *comparative support relation*. To see that this is a plausible constraint, consider what a *proto-support relation* $\succcurlyeq_\alpha$ must be like for it to fail to be *extendable* to a *complete relation*. *Extendability* is a purely syntactic requirement. That is, an *extension* $\succcurlyeq_\beta$ of a relation $\succcurlyeq_\alpha$ need not take on any of the meanings that one might have associated with the sentences of $\succcurlyeq_\alpha$. Rather, an *extension* $\succcurlyeq_\beta$ of $\succcurlyeq_\alpha$ is only required to agree with the definite comparisons between formal sentences already engendered by $\succcurlyeq_\alpha$, and to do so while continuing to satisfy the (syntactically specified) axioms 0-8. So, a *proto-support relation* $\succcurlyeq_\alpha$ can only fail to be *extendable* to a *complete relation* provided that no such *complete extension* can satisfy the syntactic restrictions embodied by axioms 0-8. In that case the *definite* strength comparisons (those of form $A|B \succ_\alpha C|D$ and $E|F \approx_\alpha G|H$) already specified by $\succcurlyeq_\alpha$ must, in conjunction with axioms 0-8, *require* that some of the remaining argument pairs remain *incomparability in strength*, not because of what their premises and conclusions *mean* (semantically), but merely to avoid implying an explicit syntactic contradiction with the *proto-support axioms*. In other words, any *proto-support relation* for which there *cannot possibly be* a *complete extension* is will already have a kind of looming incoherence due to the syntactic argument forms of its *definite* argument strength comparisons; and it can only stave off explicit incoherence (via the derivation of an explicit contradiction) by forcing some of the remaining syntactic argument pairs to remain *incomparable in strength*. While it makes good sense to declare particular argument pairs to be *incomparable in strength* when based on *what they say* (i.e. based on their semantic content) there is no appropriate basis on which to compare their strength. But a legitimate *comparative support relation* should not fix *definite strength comparisons* among some arguments, and then declare the remaining arguments *incomparable* merely to avoid syntactic incoherence. The requirement of *extendability* to *complete comparisons* simply eliminates from consideration all those *proto-support-strength relations* that deign to be so parochial.

I have already argued that requiring a *comparative support relation* to satisfy rule 10 (or 10^*) is completely innocuous. It would merely require the language of each relation to have the ability to describe various fair lotteries for finite numbers of tickets. (We can certainly do that in natural languages like English.) However, we are only requiring that *comparative support relations* satisfy the weaker constraint of being *extendable* to a language that describes lotteries (or similar equal-partitions) of this sort. This should raise no controversy. And now we have also seen why it is perfectly reasonable to require *comparative support relations* to also be *extendable* in principle to *complete relations* – i.e. to relations that satisfy rule 9.

The above representation theorem shows precisely the ways in which the *comparative support relations*, and the designated subclasses of them, are representable by *Popper functions*. Each *comparative support relation* that is itself already *complete* is *uniquely* represented by a single *Popper function*, and only the *complete* relations are *uniquely* representable in this way. Each *comparative support relation* that fails to be complete will be represented by a number of distinct *Popper functions*, and thus by a class of representing *Popper functions*, where each representing *Popper function* is an overly determinate (i.e. *complete*) model that captures the *comparative support strength relationships* embodied by the qualitative comparative relation.

Those *comparative support relations* that are not ***potentially-Archimedean*** only fail to be *extendable* to **complete-Archimedean** relations due to cases where they treat an argument $H_1|E_1$ as only *infinitesimally stronger* than another argument $H_2|E_2$. Such argument pairs are so close in comparative strength that each representing *Popper function* has to assign them identical numerical values. One might be tempted to just toss out relations of this sort, and to require that all *comparative support relations* be ***potentially-Archimedean*** *relations*. However, it turns out that some of the non-Archimedean relations capture intuitively plausible cases where some arguments should indeed be infinitesimally stronger than others. The so-called de Finetti lottery provides an example of such a case.[12]

A de Finetti-lottery-like Case:

There exist *non-Archimedean comparative support relations* $\succcurlyeq$ of the following kind:

(i) For a countably infinite set of sentences $\{H_1, ..., H_n, ...\}$ and a sentence B, $B \not\Rightarrow \sim H_i$, and (for each distinct i, j) $B \Rightarrow \sim(H_i \cdot H_j)$ and $H_i|B \approx H_j|B$.

Now notice that for any relation $\succcurlyeq$ that satisfies (i), it follows form the *proto-support axioms* that:

(ii) For each integer $n \geq 2$, for the n sentences $H_1, ..., H_n$, there is some sentence F [i.e. let F be $(B\cdot((H_1 \vee H_2) \vee ... \vee H_n))$] such that:
$F \not\Rightarrow \sim H_i$, and (for distinct i, j) $F \Rightarrow \sim(H_i \cdot H_j)$ and $H_i|F \approx H_j|F$, and $F \Rightarrow (H_1 \vee H_2 \vee ... \vee H_n)$.
(Thus, rule 10 is automatically satisfied.)

No probabilistic representation on the standard real numbers can precisely model condition (i). That is, even for *Popper functions*, given an infinite collection of alternatives $\{H_1, ..., H_n, ...\}$

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[12] De Finetti (1974) illustrates the idea that each member of an infinite collection of alternative hypotheses may be equally probable by describing a lottery with an infinite number of tickets, where each ticket is labeled with a natural number, and where exactly one randomly selected ticket will win. He uses this example to argue against the probabilistic axiom of *countable additivity*. But that's not relevant to my point here. In the following example think of $\{H_1, ..., H_n, ...\}$ as an infinite set of alternative scientific hypotheses judged equally plausible on the basis of the plausibility considerations stated by premises conjoined within statement B.

such that (for each distinct i, j) $P[(H_i \cdot H_j) \mid B] = 0$ and $P[H_i \mid B] = P[H_j \mid B]$, it follows that for all H_i, $P[H_i \mid B] = 0$ (rather than permitting $P[H_i \mid B] > 0$, as the clause '$B \not\Rightarrow \sim H_i$' in condition (i) suggests, since it implies $B|B \succ H_i|B$). Under these same circumstances one can still *use* a *Popper function* P to model condition (ii). For each finite subset of n members of $\{H_1, ..., H_n, ...\}$, where F is the sentence $(B \cdot ((H_1 \vee H_2) \vee ... \vee H_n))$, the function P described two sentences back can be permitted to allow that $P[(H_i \cdot H_j) \mid F] = 0$, $P_\alpha[H_i \mid F] = P_\alpha[H_j \mid F] > 0$, $P_\alpha[(H_1 \vee H_2 \vee ... \vee H_n) \mid F] = 1$. But this probabilistic model of condition (ii) does not *follow from* a probabilistic model (or near-model) of condition (i); whereas for the *non-Archimedean comparative support relations*, condition (ii) does indeed follow from condition (i). Thus, among the *comparative support relations*, the *Non-Archimedean relations* exhibit the ability to capture situations involving evidential support that cannot be fully represented by any of the *Popper functions*.

4. Conclusion

Bayesian confirmation theory represents evidential support for hypotheses in terms of conditional probability functions. But in real scientific contexts the strengths of plausibility arguments for various alternative hypotheses, as represented by prior probability assignments, are often vague and imprecise, and so not properly rendered by the kinds of precise numbers that conditional probability functions assign. Rather, prior probabilities are better represented by ranges of numerical values that capture the imprecision in assessments of how much more or less plausible one hypothesis is than another. Indeed, this problem of over-precision not only plagues the assignment of prior probabilities to hypotheses. In many cases the Bayesian likelihoods may also have rather vague values that are merely constrained within some bounds. This situation places the Bayesian approach to evidential support in the predicament of first proposing a probabilistic logic that assigns overly precise numerical values to argumentative support strengths, and then having to back off the over-precision by acknowledging that in many realistic applications the proper representation of evidential support should employ whole sets of conditional probability functions that represent ranges of reasonable values for the priors and the likelihoods.

The *qualitative logic of comparative support* described here provides a plausible rationale for the probabilistic Bayesian approach to evidential support. It suggests that the overly precise *probabilistic confirmation functions* are merely representational stand-ins for a deeper qualitative logic of comparative argument strengths captured by the *comparative support relations*. The *representation theorem* indicates that a *comparative support relation* will generally be *represented* by a set of conditional probability functions that spans ranges of numerical support strengths. For, the qualitative comparisons of argument strengths captured by a *comparative support relation* will often be represented equally well by a host distinct conditional probability functions. The deeper qualitative logic has the ability to exhibit the kind of incomparability in strength among argument pairs that is ubiquitous among real comparisons between arguments. Each *representing* probability function has to assign numerical support values to all pairs of statements, even though many of those numerical assignments are merely a meaningless artifact of how probability functions are defined. I suggest that in a Bayesian confirmation theory we commonly employ these overly precise *probabilistic representations* because they are computationally easier to use than the qualitative comparative support relations they represent.

But, arguably, all of the features of evidential support that we really care about are captured by the comparative relationships among argument strengths realized by the *comparative support relations* and their logic. The *probabilistic representation* of the logic of evidential support is merely a convenience, an over-determined way to represent the deeper qualitative logic of *comparative evidential support strengths*. More generally, this *logic of comparative support* provides a foundation for the notion of evidential support that underlies both classical deductive logic and probabilistic inductive logic.

Acknowledgements: I presented an early version of this paper at the *Workshop on Conditionals, Counterfactuals, and Causes in Uncertain Environments* at the University of Dusseldorf (May 20-22, 2011). Thanks to Gerhard Schurz and Matthias Unterhuber for making this workshop work so well. Thanks to all of the workshop participants for their extremely helpful questions and feedback. I owe a special debt to David Makinson, Alan Hájek, and Chris Hitchcock. Each provided comments on evolving drafts, which resulted in major revisions and improvements.

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