Invited Review

Distance-based and ad hoc consensus models in ordinal preference ranking

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Abstract

This paper examines the problem of aggregating ordinal preferences on a set of alternatives into a consensus. This problem has been the subject of study for more than two centuries and many procedures have been developed to create a compromise or consensus.

We examine a variety of structures for preference specification, and in each case review the related models for deriving a consensus. Two classes of consensus models are discussed, namely ad hoc methods, evolving primarily from parliamentary settings over the past 200 years, and distance or axiomatic-based methods. We demonstrate the levels of complexity of the various distance-based models by presenting the related mathematical programming formulations for them. We also present conditions for equivalence, that is, for yielding the same consensus ranking for some of the methods. Finally, we discuss various extensions of the basic ordinal ranking structures, paying specific attention to partial ranking, voting member weighted consensus, ranking with intensity of preference, and rank correlation methods, as alternative approaches to deriving a consensus. Suggestions for future research directions are given.

Keywords: Ranking; Ordinal preferences; Distance; Consensus; Correlation; Voters; Power indices

1. Introduction

Many real world decision problems involve the use of rank order or ordinal scale data. Among the many situations that can give rise to such data, are those commonly found in consumer survey settings, wherein
opinions as to preferences are often given in a pairwise binary choice format. When a consumer is asked, for example, to rate or compare several flavors of pudding, it is natural that an ordinal response be given: “I prefer the flavor of chocolate to vanilla”. In other situations, the consumer may be asked to rate a product on a Likert scale (e.g. a 5-point scale). Pairwise comparison information results, as well, when an optometrist requests the patient to indicate which of two lenses allows that person to see the clearest. It would be unreasonable in such situations to expect anything beyond some form of qualitative response such as these.

Similar ordinal data arise naturally in preferential election settings. Consider the situation in which each voter is requested to choose a subset of candidates from a ballot, and to rank order that subset from most to least preferred. Such a voting format is relatively common in municipal elections where a number of candidates are required to fill various positions.

Many sports events involve the direct competition of one individual or team against another. One particular form of competition, the tournament, has been the subject of extensive study over many decades. See, for example, Ali et al. [1], Goddard [25], Moon [30] and others. Chess tournaments, round-robin tennis, and other competitive sports, typify this setting. Many different approaches have been developed in the literature to score the players in a tournament. Some of these methods contain features which account for such things as player strength. See, [14].

Such ordinal preference data have been the subject of study for over two centuries. Initially, a problem arising in the theorizing on preferential elections of the 18th century, it has evolved into the social choice theory of today. An important group of problems involving ordinal data and ranking concern the aggregation of preferences, provided by a set of individuals, into a group preference function or a consensus. Numerous authors have investigated problems of ranking and consensus, including Blin [3], Cook et al. [17], Kemeny and Snell [20], and Kendall [21].

We point out here that in addition to the literature on such problems, there is, as well, a significant volume of research on the related topic of multi-criteria decision making (MCDM). Problems in the MCDM setting very often involve dealing with ordinal data of the type described above. While the emphasis in this paper is on the former problem (ranking and consensus), not the latter, we later make mention of some literature pertaining to a crossover between these two fields.

Dealing with problems involving ordinal data provided by individual responses, generally involves three issues. The first issue pertains to the format in which data concerning a set of alternatives should be collected. This depends upon the application under investigation, and the number of alternatives at hand. The second issue, in some settings, is that of resolving inconsistencies in preferences supplied by the respondent. Such inconsistencies arise when the data format chosen involves pairwise comparisons of alternatives. Finally, when multiple respondents supply preferences concerning a set of alternatives, there is the issue of combining these preferences into a consensus ranking of those alternatives.

In this paper we concentrate primarily on aggregation of ordinal preferences to form a consensus. While Section 2 looks at representation of preferences in various forms, we touch only briefly on the issue of intransitivity and its resolution. (There is an extensive literature on tournament theory where pairwise comparison, intransitivity, and the resolution thereof are the focus of discussion. We do not discuss this herein.) Sections 3 and 4 examine the main issue of consensus among ordinal rankings of a set of alternatives. We discuss two broad classes of approaches to consensus. The first, examined in Section 3, entails the various ad hoc procedures developed over time, and which have arisen primarily from parliamentary and committee settings. The second, the subject of Section 4, discusses the more formal methodologies for consensus, namely those that are based on a measure of distance of the desired consensus from the individual voter responses. Section 5 provides a form of synthesis of the various distance-based and ad hoc methodologies. In the first analysis, we examine the conditions under which the results from one type of model will be the same as under another type. In the second analysis, the distance-based methods are compared in terms of the level of complexity involved. This section also discusses some alternative formulations of the consensus problem, and looks at the KS formulation from the perspective of solution methodologies. Section 6
discusses some extensions to the basic consensus approaches, and, as well, provides some insights into the general principles surrounding consensus. In particular, we examine an emerging issue in consensus formation, namely that involving voter importance and the development of power indices. Conclusions and directions for further research are provided in Section 7.

2. Representation of preferences in ordinal data settings

Ordinal data may appear in several formats. We examine three of the most common such formats.

2.1. Object-to-object representation

A common framework for eliciting individual preferences is the pairwise comparison method, in which each pair of alternatives or objects is compared in an ordinal sense. Specifically, preferences concerning \( n \) alternatives are represented in an \( n \times n \) pairwise comparison matrix \( A = (a_{ij}) \), where

\[
a_{ij} = \begin{cases} 
1 & \text{if alternative } i \text{ is preferred to } j, \\
1/2 & \text{if } i \text{ and } j \text{ are tied,} \\
0 & \text{otherwise.}
\end{cases}
\]  

Several variations on this representation appear in the literature. In the case of 4 alternatives, \( a, b, c, d \), where \( a \) is in second place, \( b \) in first, \( c \) in fourth, and \( d \) in third, the preference matrix is given by \( A_1 \). Alternatively, if \( a \) is in first place, \( b \) in fourth, and \( c \) and \( d \) are tied for second and third positions, the representation is that given by \( A_2 \).

\[
A_1 = \begin{array}{cccc}
0 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\end{array}, \quad A_2 = \begin{array}{cccc}
0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1/2 \\
0 & 1 & 1/2 & 0 \\
\end{array}
\]

It is noted that (2.1) is equivalent to the matrix representation of Kemeny and Snell [27], where \( a_{ij} = 0 \) if \( i \) and \( j \) are tied, \( a_{ij} = 1 \) if \( i \) is preferred to \( j \), and \( a_{ij} = -1 \) if \( j \) is preferred to \( i \). From this point on, we refer to this as the KS representation.

In the KS representation, all pairs of alternatives are compared. In case a strict or strong preference cannot be expressed, KS regard this as a “tie”. No distinction is made there between this and the situation where the voter is indifferent or has no opinion. In a later section, this issue is addressed.

2.1.1. Resolving intransitivity in pairwise comparison preferences

In pairwise comparison settings, intransitivity is a common phenomenon. Tournament ranking situations give rise naturally to outcomes wherein, for example, player \( a \) defeats \( b \) who defeats \( c \), yet \( c \) defeats \( a \). In such intransitive situations the pairwise responses do not directly translate into a ranking of the alternatives involved. Numerous methods have been proposed in the literature for constructing a transitive set of preferences from intransitive responses. The “Kendall scores method” [28] and its extensions are examples of procedures designed specifically for ranking alternatives (players) in tournament structures.
In consensus settings, intransitive responses are not an issue per se, except for the fact that it is desirable that any preference structure representing a consensus among the set of such responses, should itself be transitive. We address this issue later.

2.2. Vector representation

One of the most common preference representations is the vector format $A = (a_1, a_2, \ldots, a_n)$, where $a_i$ is the rank or priority assigned to alternative $i$. This method dates back at least to Borda [6], and is the basis of the well known “method of marks”, and later the Kendall scores method, mentioned above. The vector representation of the preference relation given by $A_1$ above is $(2, 1, 4, 3)$. That is, alternative $a$ is ranked second, $b$ first, $c$ fourth and $d$ third. The priority vector corresponding to $A_2$ is $(1, 4, 2.5, 2.5)$. The 2.5 designation indicates that alternative $c$ and $d$ are tied for second and third places.

2.3. Object-to-rank representation

Blin [3] has suggested an alternative to the KS model for complete orderings, and Armstrong et al. [2] have extended this to include ties (weak orderings). Specifically, an ordering is defined by matrix $P = (p_{ij})$, where

$$p_{ij} = \begin{cases} 1 & \text{if alternative } i \text{ has rank } j, \\ 0 & \text{otherwise}. \end{cases}$$

(2.2)

It is assumed here that $i$ takes on the values $1, 1.5, 2, 2.5, \ldots, n - 1, n - 0.5, n$. The $P$-matrices corresponding to $A_1$ and $A_2$ above would then be given by

$$P_{A_1} = \begin{array}{ccccccc} 1 & 1.5 & 2 & 2.5 & 3 & 3.5 & 4 \\ a & 0 & 0 & 1 & 0 & 0 & 0 \\ b & 1 & 0 & 0 & 0 & 0 & 0 \\ c & 0 & 0 & 0 & 0 & 0 & 1 \\ d & 0 & 0 & 0 & 1 & 0 & 0 \end{array}, \quad \text{and} \quad P_{A_2} = \begin{array}{ccccccc} 1 & 1.5 & 2 & 2.5 & 3 & 3.5 & 4 \\ a & 1 & 0 & 0 & 0 & 0 & 0 \\ b & 0 & 0 & 0 & 0 & 0 & 1 \\ c & 0 & 0 & 0 & 1 & 0 & 0 \\ d & 9 & 9 & 9 & 1 & 0 & 0 \end{array}.$$

3. Consensus among ordinal preferences: Ad hoc methods

Numerous approaches have been suggested in the literature for aggregating individual rankings in order to arrive at a compromise or consensus. While some of these approaches can be linked to a particular piece of literature, e.g. [6], others have simply evolved over time via parliamentary procedures, preferential voting needs, etc. In this section we briefly examine some of these “ad hoc” approaches. These can be grouped under two headings—elimination and non-elimination methods.

3.1. Non-elimination methods of consensus

3.1.1. Borda’s method of marks

This approach, due to Borda [6], and later discussed at length by Kendall [28], is based on deriving the total of the ranks for each alternative as assigned by the voters. Consider the following 3-alternative, multivoter example:
Total for
\[ a: 23 \times 1 + 17 \times 3 + 2 \times 2 + 10 \times 2 + 8 \times 3 = 122, \]
\[ b: 23 \times 2 + 17 \times 1 + 2 \times 1 + 10 \times 3 + 8 \times 2 = 111, \]
\[ c: 23 \times 3 + 17 \times 2 + 2 \times 3 + 10 \times 1 + 8 \times 1 = 127. \]

The consensus given by Borda’s Method is then \( b > a > c \) or \( A^* = (2, 1, 3) \).

Several modifications of the Borda Method have been developed including those due to Armstrong et al. [2], and Cook et al. [9].

### 3.1.2. Simple majority rule or Condorcet’s method

Condorcet’s [8] proposed what is commonly known as the simple majority rule method, whereby alternative \( x \) should be declared the winner if for all \( y \neq x \), \( x \) is preferred to \( y \) by more voters than the number who prefer \( y \) to \( x \). Similarly, \( y \) would be ranked second if for all \( z \neq x \) or \( y \), \( y \) is preferred to \( z \) by more voters than the number who prefer \( z \) to \( y \). Consider the following example:

<table>
<thead>
<tr>
<th>Alternative</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>23 votes</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>17 votes</td>
<td>3</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2 votes</td>
<td>2</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>10 votes</td>
<td>2</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>8 votes</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

Thus, the consensus ranking by simple majority rule is \( a > b > c \) or \( A^* = (1, 2, 3) \). One problem cited by Condorcet, that is often encountered in applying this method, is the occurrence of intransitivities, giving rise to the so-called “paradox of voting” or Condorcet effect. Example 1 above illustrates this phenomenon. There, \( a > b \) by 33 voters out of 60, \( b > c \) by 42 voters, and yet \( c > a \) by 35 voters. Thus, a cycle arises, and the simple majority procedure breaks down. It has been shown that in the case of a uniform distribution of 3 alternatives, intransitivities occur 8.8% of the time. For 4 alternatives, this
probability is approximately 16%. Niemi and Weisberg [31] and others have obtained estimates of such probabilities for a number of combinations of voters and alternatives. Several “Condorcet completions” have been developed to deal with such intransitivities.

3.2. Elimination methods

These methods gained popularity in parliamentary settings.

3.2.1. Runoff from top method

One such procedure consists of each individual first voting for the prospect he/she most prefers, and if there is no majority on a first ballot, a second vote is taken after eliminating the prospect with the fewest “first choice” votes on the first ballot. This appears to be identical with what is sometimes called the “West Australian System” and very close to the so-called “English System”, particularly for only three prospects.

3.2.2. Runoff from bottom method

This approach has approximately the same appeal as the runoff from top method. On each successive ballot the voters choose the prospect to eliminate.

3.2.3. The American System

The system sometimes identified as the “American System” is apparently designed only for use with preferential ballots which collect full rankings on the first round. Here, if there is no majority of first choice votes, the option with the fewest first choices is eliminated along with all those ballots whose first choice was the eliminated option.

3.2.4. Pairwise majority rule

Finally, as described above, there is the basic method defined by “pairwise majority” rule, which has special practical as well as theoretical appeal, as long as it results in a determinate outcome. Some other methods have sought to make simple modifications to the majority rule approach [19]. In Copeland’s method, the prospect of $x$ is more preferred the greater the number of prospects which lost to $x$ relative to the number to which $x$ loses.

4. Distance-based consensus

4.1. Deriving consensus based on a set of axioms

In this section, aggregation or consensus among a set of preferences is examined from the point of view of a distance function. This concept has intuitive appeal in that a consensus is defined to be that set of preferences which is closest, in a minimum distance sense, to voter responses. This idea was first advanced by KS [27], and was later adopted by Blin [3] and by Cook and Seiford [16] (hereafter called CS).

The approach is to define a distance function on the set of all preference orders which satisfies certain desirable properties. These properties or axioms are related to social choice properties. For purposes of presentation of a common set of axioms, the CS vector model will be used as the preference representation.

CS propose that any distance function $d_{cs}$ on the set of all priority vectors should satisfy the axioms:

- Axiom 1: $d_{cs}(A, B) \geq 0$, with equality iff $A \equiv B$.
- Axiom 2: $d_{cs}(A, B) = d_{cs}(B, A)$. 

Axiom 3: \( d_{cs}(A, C) \leq d_{cs}(A, B) + d_{cs}(B, C) \) for any three rankings \( A, B, C \), with equality holding if and only if ranking \( B \) is between \( A \) and \( C \). \( A, B, \) and \( C \) and are said to lie on a line in this case, hence \( d_{cs} \) is additive on lines.

Axiom 4: (Invariance) \( d_{cs}(A, B) = d_{cs}(A', B') \), where \( A' \) and \( B' \) result from \( A \) and \( B \) respectively by the same permutation of the alternatives in each case.

Axiom 5: (Lifting from \( n \) to \( (n + 1) \)-dimensional space). If \( A^* \) and \( B^* \) result from \( A \) and \( B \) by listing the same \( (n + 1) \)st alternative in last place, then \( d_{cs}(A^*, B^*) = d_{cs}(A, B) \).

Axiom 6: (Scaling) The minimum positive distance is 1.

It can be shown that the unique distance function which satisfies this set of properties is the \( \ell^1 \) norm

\[
d_{cs}(A, B) = \sum_{i=1}^{n} |a_i - b_i|.
\]

Consensus among a set of voter priority vectors \( \{A^i\}_{i=1}^{m} \) is then given by the vector \( B^* = (b_1^*, b_2^*, \ldots, b_n^*) \) which solves the minimization problem

\[
\sum_{i=1}^{m} d_{cs}(A^i, B^*) = \min_{B} \sum_{i=1}^{m} d_{cs}(A^i, B) = \min_{B} \sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}^i - b_j|.
\]

A similar axiomatic structure was proposed by KS for pairwise comparison priorities, and for the Blin method by Armstrong et al. [2]. The KS distance is defined as

\[
d_{KS}(A, B) = \sum_{i} \sum_{j} |a_{ij} - b_{ij}|,
\]

where \( A \) and \( B \) are pairwise comparison matrices. The Blin distance function is given by

\[
d_B(P, Q) = \sum_{i} \sum_{j} |p_{ij} - q_{ij}|,
\]

where \( P, Q \) are object-to-rank binary matrices.

4.2. Preference representation and mathematical programming model structures

To better understand the distance models presented above, it is useful to examine an interesting connection that can be derived by starting with the Blin model and extending it to include degree of disagreement. Two of these extensions lead directly to the KS and CS models, respectively. Approaching the design of the latter two models from this direction lends an important insight into their levels of difficulty, vis-à-vis their solution.

4.2.1. Rank-based distance

One point of departure from the simple Blin representation is to define a function in which the aggregate disagreement between voters is measured according to the location of the alternatives, relative to the various rank positions.

Definition 4.1. The position \( j \) forward indicator vector \( P^+(j) \) and the position \( j \) backward indicator vector \( P^-(j) \) are those vectors, whose \( k \)th components are given by

\[
(P^+(j))_k = \begin{cases} 
1 & \text{if object } k \text{ is ranked in a lower position than } j, \\
0 & \text{otherwise}, 
\end{cases}
\]

and

\[
(P^-(j))_k = \begin{cases} 
0 & \text{if object } k \text{ is ranked in a lower position than } j, \\
1 & \text{otherwise}, 
\end{cases}
\]
and

\[(P^-(j))_k = \begin{cases} 
1 & \text{if object } k \text{ is ranked in a higher position than } j, \\
0 & \text{otherwise,}
\end{cases} \]

respectively.

To illustrate, consider the ranking vector (2, 3, 1, 4). That is, object #1 has rank 2, #2 has rank 3, #3 rank 1, and #4 rank 4. The \((a_{ij})\) matrix is

<table>
<thead>
<tr>
<th>Rank</th>
<th></th>
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</thead>
<tbody>
<tr>
<td>obj</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Now,

\[
P^+(1) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad P^+(2) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad P^+(3) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad P^+(4) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.
\]

That is, \(P^+(1)\) shows those objects whose rank is 2 or more, \(P^-(2)\) shows those objects whose rank is 3 or more, etc. A similar interpretation can be given for the \(P^-(j)\).

One can view the total disagreement between two rankings (as to the relative positioning of objects) as the difference between the maximum possible agreement and the actual agreement present. With this in mind, we define position or rank-based distance.

**Definition 4.2.** The rank-based distance function \(d_p(A, B)\) is given by

\[
d_p(A, B) = n(n - 1) - \sum_{j=1}^{n} [(P^+_A(j), P^+_B(j)) + (P^-_A(j), P^-_B(j))].
\]

Cook and Kress [11] prove that this rank-based distance function is equivalent to the CS distance function.

4.2.2. Linear assignment formulation of the consensus problem

The consensus ranking \(X^\tau\) is the one obtained by minimizing \(\sum_{\ell=1}^{m} d_p(A^\ell, X)\). That is, the consensus formation problem may be stated as

Max \(\sum_{\ell=1}^{m} \sum_{j=1}^{n} \sum_{i=1}^{n} \left[\left(\sum_{t=j+1}^{n} a_{it}\right)\left(\sum_{t=j+1}^{n} x_{it}\right) + \left(\sum_{t=1}^{j-1} d_{it}\right)\left(\sum_{t=1}^{j-1} x_{it}\right)\right]\)

subject to \(\sum_{i=1}^{n} x_{ij} = \sum_{j=1}^{n} x_{ij} = 1, \quad x_{ij} \geq 0 \text{ for all } i, j,\)

which is a linear assignment problem.
4.2.3. Alternative-based distance

In a fashion similar to that presented above, one can construct an alternative-based distance function.

Definition 4.3. The alternative i forward indicator vector 0+ (i) and the alternative i backward indicator vector 0- (i) are those vectors whose kth components are given by

\[(0^+(i))_k = \begin{cases} 1 & \text{if alternative } k \text{ is ranked lower than alternative } i, \\ 0 & \text{otherwise}, \end{cases}\]

and

\[(0^-(i))_k = \begin{cases} 1 & \text{if alternative } k \text{ is ranked higher than alternative } i, \\ 0 & \text{otherwise}. \end{cases}\]

Definition 4.4. The alternative-based distance function \(d_A(A, B)\) is given by

\[d_A(A, B) = n(n - 1) - \sum_{j=1}^{n} [\langle O_A^+(j), O_B^+(j) \rangle + \langle O_A^-(j), O_B^-(j) \rangle].\]

The inner product \(\langle O_A^+(j), O_B^+(j) \rangle\), for example, is simply the number of instances in which there is agreement between A and B that alternative j is preferred to some object. The alternative-based distance is, therefore, the total number of ordered pairs in which there is disagreement. Cook and Kress [11] show that the alternative-based distance \(d_A(A, B)\) is equivalent to the KS distance \(d_K(A, B)\).

4.2.4. Quadratic assignment formulation

Consensus, in the alternative-based distance format is obtained by maximizing

\[\sum_{i=1}^{n} \sum_{j=1}^{n} \langle O_A^+(i), X^+(i) \rangle,\]

such that \(X\) is a permutation matrix. Cook and Kress show that this consensus formation problem may then be formulated as the quadratic assignment problem

\[
\text{Max} \quad \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{t=j+1}^{n} \hat{a}_{ik} x_{ij} \sum_{t=j+1}^{n} x_{kt}
\]

subject to \(\sum_{i=1}^{n} x_{ij} = \sum_{j=1}^{n} x_{ij} = 1, \quad x_{ij} \geq 0 \text{ for all } i, j,\)

where

\[\hat{a}_{ik} = \sum_{r=j(i)+1}^{n} a_{ik}^r.\]

To summarize the above, it follows that the simple “agreement–disagreement” model of Blin [3] represents a type of kernel for the formation of more sophisticated and well-known measures. In terms of complexity, both the Blin and CS models can be viewed as linear assignment models (we have not shown here the Blin formulation as an assignment problem). The KS structure, however, is equivalent to a quadratic assignment model.
These mathematical programming formulations offer the opportunity to perform sensitivity analyses as well. It may, for example, be desirable to investigate the impact on the consensus of a set of rankings if certain changes happen in voter response.

4.3. Pairwise comparison distance: Alternative formulations and solution methods

Computationally efficient solution procedures for the KS distance metric have eluded researchers for decades. The need to derive a (consensus) matrix, with the requisite transitivity property, has rendered the problem rather out of reach when the number of alternatives becomes large. In a recent paper by Cook et al. [10], it is shown that the KS metric can be represented in linear integer programming format (despite the fact that it has the quadratic structure described above). Specifically, let $r_{ij}$ denote the number of voters who ranked alternatives $i$ and $j$ in the order $j > i$. Define two sets of variables $x_{ik}, y_{ij}$ where $x_{ik}$ is 1 if alternative $i$ is ranked in $k$th position, and $y_{ij} = 1$ if $i$ is preferred to $j$, and is 0 otherwise. Now solve the binary integer programming problem

$$\min \sum_i \sum_j r_{ij}y_{ij}$$

subject to

$$\sum_k x_{ik} = 1, \quad \forall i,$$

$$\sum_i x_{ik} = 1, \quad \forall k,$$

$$x_{ik} \leq \sum_{m=k+1}^N x_{jm} + (1 - y_{ij}) \quad \forall i, j, k,$$

$$x_{jk} \leq \sum_{m=k+1}^N x_{im} + y_{ij} \quad \forall i, j, k,$$

$$x_{ik}, y_{ij} \in \{0, 1\}.$$  

Because of the need to impose integrality restrictions on the variables, (unimodularity is not present), this problem still represents a severe computational hurdle when there are large numbers of alternatives. For the case where attention is restricted to strong preferences (no ties) in the final consensus ranking, a branch and bound algorithm has been developed (see [10]), and tested on a large sample of problems of varying sizes and complexities. It remains an open research issue to develop the methodology and efficient algorithms to address the case where the consensus is permitted to include weak preferences, or even partial preferences, as discussed later.

Recognizing the intractability of the KS metric, a recent paper by Emond and Mason [22] compares this metric with two rank correlation coefficients. Their proposal is to use these coefficients in place of KS, as bases for deriving a consensus among a set of ordinal rankings. One of these coefficients, Kendall’s tau, is found to be flawed in regard to the treatment of tied preferences in weak orderings. The authors present an alternative, new rank correlation approach that resolves the inherent shortcomings of Kendall’s tau. Specifically, Emond and Mason propose using the correlation coefficient

$$\tau_s(A, B) = \left( \sum_{i=1}^n \sum_{j=1}^n a_{ij}b_{ij} \right) / n(n - 1).$$

It is noted that $\tau_s$ is equivalent to Kendall’s tau for linear orderings, but for weak orderings, ties are given a score of 1 rather than 0. The authors show that this correlation coefficient is equivalent to the KS metric. The consensus derivation problem becomes that of finding a matrix $B$ that maximizes

$$\sum_i \tau_s(A', B).$$
This new method is shown to be more tractable than is true for the KS methodology. A branch and bound algorithm is presented, and extensions to weak and partial orderings are discussed.

One pragmatic option, in some circumstances, to the problem of deriving a minimum distance consensus, proposed originally by Kendall [28], and revisited recently by Chang et al. [7], is to develop an exhaustive list of all possible rank vectors, and the associated distance of each from the collection of rankings. For any chosen order vector (call it the norm), Chang et al. define this distance as the concordant order ratio $P/(P + Q)$, where $P$ and $Q$ are respectively the numbers of concordant and discordant pairs of alternatives, when comparing the norm to the supplied rank order vectors. All possible norm vectors are then ranked from highest to lowest in terms of their concordant order ratios. This has the advantage of providing a variety of good rank orders from which the decision maker may choose. In some cases, the optimal order may not be feasible to use, hence knowing the second or third best can aid in effective decision making.

5. Agreement among consensus methods

An issue which a number of authors have addressed has to do with the likelihood of different criteria giving rise to the same outcome (same winner or same consensus ranking). Fishburn [23], for example, has carried out a simulation study comparing Borda’s method with that of Copeland. In this particular study various combinations $(n, m)$ were examined ($n = \#$ voters and $m = \#$ alternatives) from $n = 3–21$ and $m = 3–9$. For each such combination, 1000 cases were generated. A uniform distribution of ranked votes was assumed in carrying out the simulations. In comparing the two consensus methods, the issue was whether the winning candidates matched (2nd, 3rd, …, etc. place standings were not compared). Fishburn has found, for example, in the case of 21 voters and 3 alternatives, that in 81.8% of the 1000 cases examined, all winners via Borda (i.e. the set of alternatives tied for 1st place) were also the winners in Copeland and vice versa. In 12.8% of the cases at least some Borda winner matched a Copeland winner (but not all winners under Borda matched all Copeland winters). Finally, in the remaining 5.4% of the 1000 cases, no Borda winner was a Copeland winner.

5.1. The 3-alternative case

As an illustration, in the 3-alternative case, the six possible linear ordering of the 3 alternatives are assumed to follow a Dirichlet distribution. This single peaked distribution is particularly instructive in that there is a relative independence among options. At the same time, it is sufficiently general to permit a number of possible shapes.

For the 3-alternative case, and under the assumption of Dirichlet-distributed rankings, it can be shown that the pairwise majority rule model always results in a transitive ranking. Thus, Copeland and majority rule are equivalent. It can also be shown that all of the aforementioned runoff or elimination methods will yield the same consensus ranking as well.

5.2. The general case

Cook et al. [17] examine the general case for $n$ alternatives. For this case, let $\{R^i\}_{i=1}^n$ denote the space of ordinal rankings, and $\{P^i\}_{i=1}^n$ the corresponding proportions of voters (probabilities of the $n!$ rankings being chosen). Further, define the index set $M_{ij} = \{\ell | a^\ell_i < a^\ell_j\}$. Here, $a^\ell_i$ denotes the position assigned to alternative $i$ by the $\ell$th voter. That is $M_{ij}$ is the set of all rankings in which alternative $i$ is preferred to alternative $j$. Note that $M_{ij} \cap M_{ji} = \emptyset$ and $M_{ij} \cup M_{ji} = \{1, 2, \ldots, n!\}$. With this notation, voter responses are
said to be transitive if there exists an ordering of the \( n \) alternatives (assume this is say the natural ordering \((1, 2, \ldots, n)\)) such that
\[
\sum_{i \in M_{ij}} P^i > \sum_{i \in M_{ji}} P^i \quad \text{for all } i, j, \text{ where } i < j.
\]

Clearly, if such a transitivity property holds, then the consensus under pairwise majority rule (and Copeland’s model) is the natural ordering
\[
(r_1, r_2, \ldots, r_n) = (1, 2, \ldots, n).
\]

In order to evaluate the Borda model, it is necessary to define another form of transitivity. Partition the index set \( M_{ij} \) into a series of different sets or levels. Formally, we define for alternatives \( i \) and \( j \) \((i \neq j)\) the set of difference \( c \) (or level \( c \)) by
\[
L_c = \{ \ell \mid d^i_\ell - d^j_\ell = c \}.
\]

Note that \( M_{ij} = \bigcup_{c=1}^{n-1} L_c \) and \( M_{ji} = \bigcup_{c=1}^{n-1} L_c \).

**Definition 5.1.** Voter responses are said to be weighted transitive if there exists an ordering of the \( n \) objects such that
\[
\sum_{c=1}^{n-1} c \sum_{i \in L_{(c-1)}} P^i > \sum_{c=1}^{n-1} c \sum_{i \in L_c} P^i \quad \text{for all } i, j \text{ where } i < j.
\]

**Theorem 5.1.** Under the condition that voter responses follow a Dirichlet distribution, transitivity on levels is always present, and Simple Majority Rule, Borda’s method and Copeland’s method are equivalent.

### 6. Extensions

#### 6.1. Partial orderings

The Kemeny and Snell [27] representation for ordinal ranking is designed for complete and weak orderings. Specifically, each pair of alternatives is compared, but tied preferences are omitted. Bogart [4,5] extended this idea in some respects by allowing for the case where only members of a subset of the alternatives are compared, one against the others. This amounts to a partial ordering of the alternatives. In Bogart’s representation, however, there are still only two possibilities, namely, there is either a strict preference of one alternative over another or the two alternatives are not compared.

Cook et al. [15] extended the ideas of KS and of Bogart, to permit all three levels of comparison, namely, strict preferences, ties and ‘no comparison’. In this case, they use a bimatrix representation. Specifically, a partial ranking \( A \) is represented by the pair of matrices \((I, P)\), \( I = (I_{ij}) \) is referred to as the information matrix, and \( P = (P_{ij}) \) the preference matrix. Specifically, define
\[
I_{ij} = \begin{cases} 
1 & \text{if } i \text{ and } j \text{ are compared (strict preference or tie)}, \\
0 & \text{if } i \text{ and } j \text{ are not compared}, 
\end{cases}
\]
\[
P_{ij} = \begin{cases} 
1 & \text{if } i > j, \\
0 & \text{otherwise}.
\end{cases}
\]
Using a set of axioms similar to those described above, the unique distance function on the space of all partial orderings is given by

\[
d_{IP}(A, B) = \sum_{i=1}^{n} \sum_{j=1}^{n} \left[ \frac{1}{2} |I_{ij}^A - I_{ij}^B| + \left| P_{ij}^A - P_{ij}^B \right| \right].
\]

The authors show that in the subspace of all weak orderings, \(d_{IP}\) and \(d_{KS}\) are equivalent. The consensus ranking among a set of partial orderings is that ranking \(^\wedge A\) that minimizes the total absolute distance

\[
\sum_{\ell=1}^{m} d_{IP}(A^\ell, A) = \sum_{\ell=1}^{m} \sum_{i=1}^{n} \sum_{j=1}^{n} \left[ \frac{1}{2} |I_{ij}^{A^\ell} - I_{ij}^A| + \left| P_{ij}^{A^\ell} - P_{ij}^A \right| \right],
\]

where the minimum is over all partial orders \(A\). The study of partial orderings is well developed in MCDM settings as well. See, for example, [34].

Roy and Slowinski [37] have proposed a set of logical and significance conditions (as alternatives to KS and others), to support the construction of a proper distance measure between pairs of binary relations that link pairs of alternatives. This distance measure is then used in a multi-criteria analysis of water supply system programs. These authors did not, however, tackle the aggregation problem. Martel and Ben Khelifa [29] have suggested a similar distance measure to that of Roy and Slowinski, and have used it in an algorithm for determining a collective or consensus ranking from a set of partial rankings. Jabeur et al. [26], propose an extension to the Roy and Slowinski measure. Both of these measures satisfy the symmetry and triangle inequality axioms. Explicit specifications of the axiomatic structures can be found in the aforementioned literature, and will not be presented herein.

It is important to point out here that in the context of partial orderings, there is disagreement as to the “degree of incomparability” involving a pair of alternatives. Some authors (e.g. [35,39]), differentiate between the idea of “indifference” versus “incomparability” regarding the pair. Cook et al. [15], Roy and Slowinski [37], Jabeur et al. [26], and others do not make this degree of distinction.

6.2. Voter power

In the group decision making literature, particularly in ordinal ranking problems, the notion of voter power or relative importance has been largely ignored. It is typically assumed that all “voting” members have the same importance. However, it remains that in many real life settings, voters have recognized abilities and attributes, or privileged positions of power. If one had tangible estimate of member or voter weights, these members become rather like criteria in an MCDM problem. Thus, it is the weighted consensus that one seeks to derive. Emond and Mason [22], for example, present their consensus models from a voter-weighted perspective.

The question then is not how to use weights in the consensus ranking problem, but rather, how to quantify these in the first instance. In some settings, the “weight” of a voter may be well defined. For example, the number of votes held by a member of parliament, or the GNP or population size of a country represented by the member on an international committee, can immediately be use as weights. However, where it is difficult to reach agreement on such measures, it is necessary to resort to some objective method to determine the values for these weights or power indices.

Game theory has been applied in some situations to measure the power of a group of members. See Deegan and Packel [20], and Turnovec [41]. Ramanathan and Ganesh [32] apply a participative approach by using an AHP [38] procedure. In this approach, each member is asked to rate, on an AHP scale, their own strength and that of the other members. One then uses the eigenvector method to derive a weight vector.

Martel and Ben Khelifa [29] propose a method to determine the relative importance of group members, by using individual outranking indices. The method is based upon the idea that the member who expresses a
strict or strong preference of one alternative over another, should be credited with more influence in dictating the collective (consensus) opinion than the member who is indifferent, unable or does not want to express his/her preferences. These authors then propose choosing weights for the members based upon the cardinal properties of their outranking indices.

Jabeur et al. [26] propose a procedure based on Zeleny [42] to determine the relative importance coefficient (r.i.c.) of each member. They first use the method of Simos [40], revised by Roy and Figueira [36], and then the DeGroot [21] method, to build for each member, a subjective component. Then, for each member and each pair of alternatives, an objective component is obtained by imposing priority rules on the preference structure. These two components are finally combined to obtain the r.i.c. for each member and each pair of alternatives. A full discussion is found in Jabeur et al. [26].

6.3. Preference strength

A line of research developed in Cook and Kress [12,13], examines the measurement of preference intensity or strength. This attempts to credit preference specifications of the KS type with something beyond pure ordinal comparison of pairs of alternatives. Specifically, the voter is permitted to express intensity of preference.

In the KS framework the voter ranks a set of objects without any expression of degree of preference. For example, consider 5 objects \( a_1, a_2, a_3, a_4, a_5 \) ranked as follows:

\[
A = \begin{bmatrix}
a_1 \\
a_3 \\
a_2 \\
a_4 \\
a_5
\end{bmatrix}.
\]

That is, object \( a_1 \) is in first place, \( a_3 \) and \( a_5 \) are tied for second place, \( a_2 \) is in third place and \( a_4 \) is in fourth place. The KS representation of this object listing in matrix form as follows:

<table>
<thead>
<tr>
<th></th>
<th>( a_1 )</th>
<th>( a_2 )</th>
<th>( a_3 )</th>
<th>( a_4 )</th>
<th>( a_5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_1 )</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( a_2 )</td>
<td>-1</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>( a_3 )</td>
<td>-1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( a_4 )</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>( a_5 )</td>
<td>-1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Suppose \( n \) objects are to be ranked, and \( h \) “slots” or positions are available. For example, suppose 5 objects are to be assigned to 7 available slots. Then, the above voter might now represent his/her preferences (including intensity of preference) as given by \( \tilde{A}_1 \). Alternatively, another voter who would have supplied the same preference ordering \( A \), might express the intensity as in \( \tilde{A}_2 \).

\[
\tilde{A}_1 = \begin{bmatrix}
a_1 \\
- \\
a_3, a_5 \\
a_2 \\
- \\
a_4 \\
-
\end{bmatrix}, \quad \tilde{A}_2 = \begin{bmatrix}
a_1 \\
a_3, a_5 \\
- \\
a_2 \\
- \\
a_4 \\
-
\end{bmatrix}.
\]
To incorporate both preference and degree thereof, we replace the usual binary matrix employed in the KS framework by a preference intensity matrix.

**Definition 6.1.** For any ordinal ranking \(A\), the *preference intensity* matrix \(P^A = (p^A_{ij})\) is that matrix for which \(p^A_{ij}\) is the number of positions by which \(a_i\) is preferred to \(a_j\).

As an example, the preference intensity matrix for the ranking \(A_1\) is

\[
p^A_1 = \begin{pmatrix}
a_1 & 0 & 3 & 2 & 5 & 2 \\
a_2 & -3 & 0 & -1 & 2 & -1 \\
a_3 & -2 & 1 & 0 & 3 & 0 \\
a_4 & -5 & -2 & -3 & 0 & -3 \\
a_5 & -2 & 1 & 0 & 3 & 0 \\
\end{pmatrix}
\]

Under a natural set of axioms, Cook and Kress [12] show that the unique distance function on the space of all rankings with intensity of preference is

\[
d^P = \frac{1}{2} \sum_{ij} |p^A_{ij} - p^B_{ij}|,
\]

and the *consensus ranking* \(\hat{B}\) is that ranking for which

\[
M(\hat{B}) = \sum_{\ell=1}^m d(A_\ell, \hat{B}) = \min_{B \in \beta} \sum_{\ell=1}^m d(A_\ell, B),
\]

where \(\beta\) is the set of all \(n \times n\) preference intensity matrices.

It can be shown that this problem is equivalent to a “Special Order Network” (SON) as discussed in Glover et al. [24].

### 7. Conclusions and future directions

This paper has examined the issue of aggregating ordinal voter preferences across a set of alternatives into a ranking that is meant to represent a consensus among the expressed voter responses. Ordinal data arises naturally in settings such as market research, and preferential voting in parliamentary settings. Many different approaches have been suggested for deriving a consensus, and have been the subject of extensive research as far back as 1781 [6], and possibly earlier. An important subject area, which has attracted little attention is that involving conditions under which the various methods will yield the same consensus ranking. Richelson [33], Fishburn [23], and Cook et al. [17] are among the few studies carried out. It is believed that more research is needed in this area. As well, further work on measuring voter power, along the lines of Jabeur et al. [26], will serve to enhance the applicability of the many ordinal ranking methodologies.

The study of ordinal preferences and consensus touches on a variety of fields, including tournament theory, multiple criteria decision modeling, and more recently, data envelopment analysis (DEA) involving qualitative data. In terms of the latter, it is assumed that when qualitative data for a variable are present for each decision making unit, such data result from the opinion of one voter. Multiple voter responses, and consensus among those responses have not been examined in the DEA context. This applies as well to the relatively new area of research in DEA referred to as IDEA (Imprecise DEA) by Cooper et al. [18]. Research into the handling of multiple respondent opinions in this performance measurement setting, can be an important contribution to the field.
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References


