

How to Express Self-Referential Probability*

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Abstract

We present a semantics for a language which includes sentences that can talk about their own probabilities. This semantics applies a fixed point construction to possible world style structures. One feature of the construction is that some sentences only have their probability given as a range of values. We develop a corresponding axiomatic theory and show by a canonical model construction that it is complete in the presence of the ω -rule. By considering this semantics we argue that principles such as introspection, which lead to paradoxical contradictions, should be expressed by using a truth predicate to do the job of quotation and disquotation and observe that in the case of introspection the principle is then consistent.

1 Introduction

We are interested in languages that include sentences which can talk about their own probabilities. In such languages contradictions can arise between seemingly harmless principles such as probabilism and introspection. The sentence that is used to display this is:

The probability of π is less than $1/2$. (π)

Caie (Caie, 2013) has recently used this as a *prima facie* argument against probabilism. A very natural response to this is to argue that the contradiction is only due to self-reference and should be ignored. One way to account for this intuition is to prevent such sentences from appearing in a language. However we shall argue that the result of doing this is that one cannot properly represent quantification or formalise many natural language assertions so we think that this is the wrong path to take. Instead we shall argue that such self-referential probability assertions should be expressible but one should give a careful semantics so as to avoid such contradictions to the extent that is possible. We believe such considerations will become relevant in disciplines such

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as formal epistemology and philosophy of science which use probabilistic methods as these disciplines start to work with formal languages which are more expressive.

So why should such self-referential probability sentences be expressible? Firstly, we want to have languages which can talk about probability. We do this by ensuring that they can formalise expressions such as

The probability of φ is $\geq \alpha$

In doing this we assign probabilities to sentences instead of to events which are subsets of a sample space. Although this is uncommon in mathematical study of probability it is not uncommon in logical and philosophical work and it will allow us to present the syntax of our language, which can deal with embeddings of probabilities, without first developing a semantics.

Being able to express embeddings of probabilities is useful to express agents beliefs about other agents' beliefs or generally relationships between different notions of probability. These would lead to constructions such as

$$P^A(\dots P^B(\dots P^A \dots)\dots)\dots$$

We furthermore allow for self-applied probability notions, or higher order probabilities, namely constructions such as

$$P^A(\dots P^A \dots)\dots$$

These offer us two advantages. Firstly they allow for a systematic syntax once one wishes to allow for embedded probabilities. Secondly their inclusion might be fruitful, as was discussed in (Skyrms, 1980). For example we can then represent introspective abilities of an agent or the uncertainty or vagueness about the first order probabilities. If one disagrees and wishes to argue that they are trivial and collapse to the first level then one should still not prevent them being expressed in the language but should instead include an extra principle to state this triviality of the higher levels such as adding an introspection principle which is something formalising:

If the probability of φ is $\geq \alpha$,
then the probability of “The probability of φ is $\geq \alpha$ ” is 1

In fact once we have languages which can express self-referential probabilities we see that this introspection principle along with the analogous negative introspection principle cannot be satisfied¹. This suggests that the triviality of the higher levels of probability is more substantial an assumption than it seems at first sight. These higher order probabilities are therefore at least something we should be able to express and consider.

There are two ways to formalise languages of higher order probabilities without allowing for self-referential probabilities. The first is to consider a hierarchy

¹If the probability notion satisfies the axioms of probability.

of languages. This is given by a language L_0 which cannot talk about probabilities at all, together with a metalanguage L_1 which can talk about probabilities of the sentences of L_0 , together with another metalanguage L_2 which can talk about the probabilities of sentences of L_1 etc. This leads to a sequence of language L_0, L_1, L_2, \dots each talking about probabilities of the previous language. Natural language does contain multiple probability notions, such as objective chance and the degrees of beliefs of agents, but the different notions should be able to apply to all the sentences of our language and not have a hierarchy of objective chance notions ch_0, ch_1, \dots applying to the different levels of language.

The second approach is to instead consider one language where the probability notion is formalised by an operator. This is the approach taken in (Aumann, 1999), (Halpern, 2003), (Ognjanović & Rašković, 1996) and (Bacchus, 1990) amongst others. Each of these differs in their exact set-up but the idea is that one adds a recursive clause saying: if φ is a sentence of the language then we can form another sentence of the language which talks about the probability of φ . For example one adds the clause

$$\text{If } \varphi \in L \text{ then } P_{\geq \alpha}(\varphi) \in L$$

to the construction rules of the language L . In this language $P_{\geq \alpha}$ acts syntactically like \neg instead of like a predicate so this is not a language of first order logic.

Both the typed and operator languages do not allow for self-referential probabilities but they also cannot easily account for quantification over the sentences of the language. So for example they cannot express:

Annie is certain that Billy has non-extremal degrees of belief.

There is an alternative language for reasoning about probabilities: one can add the probability notion as a standard predicate or function symbol in first order logic. In our language we will be able to express the above by:

$$P_{=}^A (\ulcorner \exists x (P_{>}^B(x, \ulcorner 0 \urcorner) \wedge P_{<}^B(x, \ulcorner 1 \urcorner)) \urcorner, \ulcorner 1 \urcorner)$$

If one takes a background of arithmetic one can derive the diagonal lemma for this language and therefore result in admitting sentences which talk about their own probabilities. Such self-referential probabilities therefore arise when we consider languages which can express such quantification.

Working with languages which can express self-referential probabilities can also be an advantage. In natural language we can assert sentences which are self-referential or not depending on the empirical situation and an appropriate formal language representing natural language should be able to do this too.

Consider the following example.² Suppose that Smith and Jones have applied for a certain job. And suppose that Smith believes that Jones is the man who will get the job and that Smith doesn't have high credence in anything which Jones says. Smith might therefore say:

²A modification of the traditional Gettier example against the Justified True Belief analysis of Knowledge

- (1) “I don’t have high credence in anything that the man who will get the job says.”

Imagine, further, that unknown to Smith, he himself, not Jones, will get the job. (1) therefore expresses a self-referential probability assertion analogous to π . To reject Smith’s assertion of (1) as formalisable would put serious restrictions on the natural language sentence which are formalisable.

We therefore accept such sentences into our formal language and we try to present a semantics for the language.

Such discussions are not new. The possibility of self-reference is also at the heart of the liar paradox. Namely a sentence which says of itself that it is not true. This can be expressed by:

$$\neg T \ulcorner \lambda \urcorner \tag{\lambda}$$

This liar sentence leads to paradox under certain basic assumptions about T , namely the principles $T \ulcorner \varphi \urcorner \leftrightarrow \varphi$ for each φ .

In Kripke’s seminal paper, (Kripke, 1975), Kripke says:

“Many, probably most, of our ordinary assertions about truth or falsity are liable, if our empirical facts are extremely unfavourable, to exhibit paradoxical features... it would be fruitless to look for an intrinsic criterion that will enable us to sieve out—as meaningless, or ill-formed—those sentences which lead to paradox.”

So if we wish our formal language to represent our ordinary assertions about probability we should allow the possibility of self-referential sentences. We should then provide a clear syntax and semantics which can appropriately deal with these sentences.

In Kripke’s paper he presents an outline of a construction of a semantics for a language with a truth predicate which he argues appropriately deals with the liar sentence. The construction formalises a procedure of evaluating a sentence. He uses three-valued logics to build up a classical semantics. The difference to a fully classical approach is that for certain sentences, such as λ , one will have both $\neg T \ulcorner \varphi \urcorner$ and $\neg T \ulcorner \neg \varphi \urcorner$. In this paper we shall present a generalisation of his semantics to also account for probability predicates. The final semantics will be classical, but for some sentences we will only assign a range of probability values not a particular probability value. So we might have $\neg P_{>}(\ulcorner \varphi \urcorner, \ulcorner 0 \urcorner)$ and $\neg P_{<}(\ulcorner \varphi \urcorner, \ulcorner 1 \urcorner)$, but only $P_{\geq}(\ulcorner \varphi \urcorner, \ulcorner 0 \urcorner)$ and $P_{\leq}(\ulcorner \varphi \urcorner, \ulcorner 1 \urcorner)$. Our generalisation follows ideas presented in (Halbach & Welch, 2009) where Halbach and Welch develop a semantics for necessity conceived of as a predicate by applying Kripke’s construction to possible world structures. In (Stern, 2014b), Stern further considers the Halbach and Welch construction from an axiomatic point of view. We use his observations to aid our development of an axiomatic system for our construction. In this paper and its sister paper (Stern, 2014a) Stern argues that when stating principles about necessity the job of quotation and disquotation should be done by a truth predicate. We argue for the same thing

here: we argue that principles such as the introspection principles are properly expressed by using the truth predicate, in our language this will be written as:

$$\top \ulcorner P_{\geq} (\ulcorner \varphi \urcorner, \ulcorner \alpha \urcorner) \urcorner \rightarrow P_{=} (\ulcorner P_{\geq} (\ulcorner \varphi \urcorner, \ulcorner \alpha \urcorner) \urcorner, \ulcorner 1 \urcorner)$$

In conclusion we believe that in light of such challenges facing languages with self-referential probabilities one should develop a careful semantics. The semantics which we construct here involves a certain amount of non-classical logic, in particular the probability notion should give only ranges of probabilities to the problematic sentences, though the final semantics is a fully classical one. Our semantics is constructed using possible world style structures therefore allowing one to use the technical advantages of these structures which were thought to only be applicable to the operator languages. Finally, we shall argue that principles such as introspection should be restated in such a framework by appealing to the truth predicate.

Before presenting the structure of our paper we will briefly mention the previous work on self-referential probabilities. Although there is not space to give proper discussion of these papers we will sketch the ideas so it is clearer where our work fits in. Firstly, in (Leitgeb, 2012) Leitgeb develops the beginnings of what might be called a revision semantics for probability, though he only goes to stage ω . He also presents a corresponding axiomatic theory. Revision semantics for truth is a popular alternative to Kripke's semantics which is the semantics which we shall focus on. Our paper can therefore be seen to connect and complement Leitgeb's work by seeing how the variety of theories of truth can lead to theories of probability. In (Caie, 2013) and (Caie, 2014) Caie argues that traditional arguments for probabilism, such as the argument from accuracy, the Dutch Book argument and the argument from calibration, all need to be modified in the presence of self-referential probabilities and that so modified they do not lead to the rational requirement for beliefs to be probabilistic. Further analysis of Caie's modifications can be found in (Campbell-Moore, n.d.). In (Caie, 2013) Caie also presents a *prima facie* argument against probabilism by noticing its inconsistency with introspection when such self-reference is around. Our proposal in this paper that introspection should be stated by appeal to a truth predicate can be seen as another response to Caie's argument. Lastly, the unpublished paper by (Christiano, Yudkowsky, Herresho, & Barasz, n.d.) also considers the challenge that probabilism is inconsistent with introspection. In this paper Christiano et al. show that probabilism is consistent with an approximate version of introspection where one can only apply introspection to open intervals of values in which the probability lies. These authors come from a computer science background and believe that these self-referential probabilities might have a role to play in the development of artificial intelligence. Both Christiano et al. and Leitgeb cannot satisfy the Gaifman condition:

$$P(\forall x \varphi(x)) = \lim_{n \rightarrow \infty} P(\varphi(\bar{1}) \wedge \dots \wedge \varphi(\bar{n}))$$

Both at the ω stage of Leitgeb's construction and in the construction by Christiano et al. there is a sentence $\varphi(x)$ such that $P(\forall x \varphi(x)) = 0$ but for each n

$P(\varphi(\bar{0}) \wedge \dots \wedge \varphi(\bar{n})) = 1$. In Section 3.2 we will show that our semantics allows P to satisfy the version of the Gaifman condition which is appropriate in our framework. We therefore do not face the same challenge.

The rest of the paper is structured as follows. In the next section, Section 2, we shall introduce the language and the defined semantics. We shall focus on a single agent but our semantics can easily generalise to multiple agents. We in fact believe that it is the multiple agent version that is of the most philosophical interest but including this would merely complicate the presentation.

In Section 3 we first show that the appropriate way to express the introspection principles in this construction is by using a truth predicate. This allows one to avoid inconsistency and is well-motivated in this semantic construction. In this section we also explain the observation that this construction can easily accommodate for σ -additive probabilities.

In Section 4 we shall give an axiomatic theory in which is intended to capture the semantics. Such a theory is important because it allows one to reason about the semantics. As was discussed in (Aumann, 1999), when one gives a possible worlds framework to formalise a game theory context the question arises of what the players know about the framework itself and this question is best answered by providing a corresponding syntactic approach. Our theory is sound, but is only complete in the presence of the ω -rule which allows one to conclude $\forall x\varphi(x)$ from all the instances of $\varphi(\bar{n})$. This is needed to fix the standard model of arithmetic. To show the completeness when the ω -rule is present we construct a canonical model. This axiomatisation is substantially new research.

Finally we finish the paper with some conclusions in Section 5.

2 A Semantics for Languages with Self-Referential Probabilities

2.1 Setup: The Language and Notation

The syntax of the language which we work with will be as follows:

Definition 2.1. Let L be some language extending the language of Peano Arithmetic. We allow for the addition of contingent vocabulary but for technical ease we shall only allow contingent relation symbols and propositional variables and not function symbols or constants.³

Let $L_{P,T}$ extend this language by adding a unary predicate T and a binary predicate P_{\geq} .

We could consider languages with multiple probability notions, then we would add the binary predicate P_{\geq}^A for each notion of probability, or agent A . But our constructions will immediately generalise to the multiple probability languages so we just focus on the language with one probability notion. We

³This restriction could be dropped, but then $t^{\mathbb{N}}$, as will be defined in Definition 2.2 would be ill-defined and this would just complicate the presentation of the material.

have included the truth predicate since it is easy to extend the definition of the semantics to deal with truth as well as probability and it is nice to see that the construction can give a joint theory of truth and probability. Additionally we shall rely on the truth predicate for our later axiomatisation and for expressing principles such as introspection.

We need to be able to represent sentences of $L_{P,T}$ as objects in the language. To this end we assume some standard Gödel coding of the $L_{P,T}$ expressions into the natural numbers. We shall also assume a coding of rational numbers into the object language since the arithmetic background makes certain things easier and we wish to have the ability also to quantify over this component. We therefore have sentences such as $P_{\geq}(\ulcorner\varphi\urcorner, \ulcorner\alpha\urcorner)$ whose intended interpretation is

“The probability of φ is $\geq \alpha$.”

Notation 2.2. ⁴The numeral of n is denoted \bar{n} and it corresponds to the expression $S(\dots S(0)\dots)$.

We assume some coding $\# : \text{Form}(L_{P,T}) \cup \mathbb{Q} \rightarrow \mathbb{N}$ which is recursive and one-to-one. For $\varphi \in \text{Form}(L_{P,T})$ and $\alpha \in \mathbb{Q}$, we let $\ulcorner\varphi\urcorner$ and $\ulcorner\alpha\urcorner$ denote the numerals corresponding to $\# \varphi$ or $\# \alpha$ respectively. We use $\text{rat}(n)$ to denote the rational number whose code is n . So $\text{rat}(\ulcorner\alpha\urcorner) = \alpha$.

We define $\neg t$ for t a term in the language to represent the syntactic operation of negating a sentence. So $\neg \ulcorner\varphi\urcorner = \ulcorner\neg\varphi\urcorner$ is a true sentence of the language. Similarly we use 1_{-} to represent “1-”, so $1_{-} \ulcorner\alpha\urcorner = \ulcorner1 - \alpha\urcorner$ is true. We also use \succ to denote the ordering $>$ on rational numbers, so $\ulcorner\alpha\urcorner \succ \ulcorner\beta\urcorner$ is true iff $\alpha > \beta$.

Finally, we denote the interpretation of the term t in \mathbb{N} by $t^{\mathbb{N}}$, for example $\bar{n}^{\mathbb{N}} = n$, and $S\bar{n}^{\mathbb{N}} = n + 1$. This is well-defined because we assumed there were no contingent function symbols in the language L .

We now introduce the other probability predicates which we use as abbreviations.

Definition 2.3. Define for terms t and s the following abbreviations:

- $P_{>}(s, t) := \exists a \succ t (P_{\geq}(s, a))$
- $P_{\leq}(s, t) := P_{\geq}(\neg s, 1_{-}t)$
- $P_{<}(s, t) := P_{>}(\neg s, 1_{-}t)$
- $P_{=}(s, t) := P_{\geq}(s, t) \wedge P_{\leq}(s, t)$

In a model which interprets the arithmetic vocabulary by the standard model of arithmetic we will have that $P_{>}(\ulcorner\varphi\urcorner, \ulcorner\alpha\urcorner)$ holds if and only if there is some $\beta > \alpha$ such that $P_{\geq}(\ulcorner\varphi\urcorner, \ulcorner\beta\urcorner)$ holds.

⁴We follow (Halbach, 2011) in much of this, so that should be referred to for additional details

2.2 The Construction of the Semantics

Kripke’s construction in (Kripke, 1975) is motivated by the idea that one should consider the process of evaluating of a sentence to determine which sentences can unproblematically be given a truth value.

To evaluate the sentence $\top 0 = 0^\top$ one first has to evaluate the sentence $0 = 0$. Since $0 = 0$ does not mention the concept of truth it can easily be evaluated. Kripke formalises this process of evaluating sentences. We shall say evaluated positively (and evaluated negatively) instead of evaluated as true (and evaluated as false) to make it clear that this is happening at the meta-level. At each stage only some sentences have been evaluated positively or negatively. So for example at the first stage $0 = 0$ is evaluated positively and $0 = 1$ is evaluated negatively but the liar sentence

$$\neg \top \lambda^\top \quad (\lambda)$$

won’t be evaluated as either. So at the second stage we can now evaluate $\top 0 = 0^\top$ positively and $\top 0 = 1^\top$ negatively, but $\top \lambda^\top$ still won’t be evaluated either way. Instead of building up the set of sentences which are evaluated positively and those that are evaluated negatively we can just focus on the positive ones and observe that φ is evaluated negatively iff $\neg\varphi$ is evaluated positively.

Since a sentence like $\top \lambda^\top$ won’t be evaluated either positively or negatively we have to give some rules for saying whether a sentence such as $\top \lambda^\top \vee \neg \top \lambda^\top$ should be evaluated positively, negatively or neither. To do this we need to provide a three-valued evaluation scheme. The scheme that Kripke focuses on is Kleene’s strong three-valued logic. The advantage of this scheme, unlike for example a scheme based on supervaluational logic, is that it is truth-functional, i.e. the evaluation of $\varphi \vee \psi$ depends only on how φ and ψ have been evaluated. As a result our semantics lends itself to axiomatic methods.

Now we consider how to add probability notions to this construction. To evaluate $\mathbb{P}_{\geq}(\top \varphi^\top, \top \alpha^\top)$ we first need to evaluate φ not only in the actual state of affairs but also in some other states of affairs. We therefore base our construction on structures with multiple possible worlds and we evaluate the sentences at all the worlds. We will assume that each world has a “degree of accessibility” relation to the other worlds which will be used to give us the interpretation of \mathbb{P}_{\geq} .

Definition 2.4 (Probabilistic Modal Structure). *A probabilistic modal structure for a language L is given by a frame and an interpretation:*

A *frame* is some $(W, \{m_w | w \in W\})$ where W is some non-empty set, we shall call its objects *worlds*, and m_w is some finitely additive measure function over the powerset of W^5 , i.e. $m_w : \mathcal{P}(W) \rightarrow [0, 1]$ satisfying

- $m_w(W) = 1$

⁵ Assuming that this is defined on the whole powerset does not in fact lead to any additional restriction since a finitely additive measure function on some Boolean algebra can always be extended to one defined on the whole powerset.

- $m_w(A) \geq 0$ for all $A \subseteq W$
- For $A, B \subseteq W$, if $A \cap B = \emptyset$ then $m_w(A \cup B) = m_w(A) + m_w(B)$

An *interpretation* \mathbb{M} assigns to each world w , an L -model $\mathbb{M}(w)$. For the purpose of this paper we assume that each $\mathbb{M}(w)$ is a standard model of arithmetic on the arithmetic vocabulary.⁶

Such structures are very closely related to type spaces which are of fundamental importance in game theory and economics. In a type space it is almost always assumed that each m_w is σ -additive. Furthermore it is often assumed that $m_w\{v | m_v = m_w\} = 1$, i.e. the agents are fully aware of their own beliefs, though these are called Harsanyi type spaces following Harsanyi's development of them in (Harsanyi, 1967). We use a different name to make it clear that we do not use these assumptions.

Suppose we have an urn filled with 9 balls, 3 of which are yellow, 3 red and 3 blue. Suppose that a random ball is drawn from the urn and the agent is told whether it is yellow or not. If a red ball has been drawn then her belief that the ball drawn is blue will be $1/2$. We represent this and her other beliefs in the probabilistic modal structure depicted in Fig. 1.

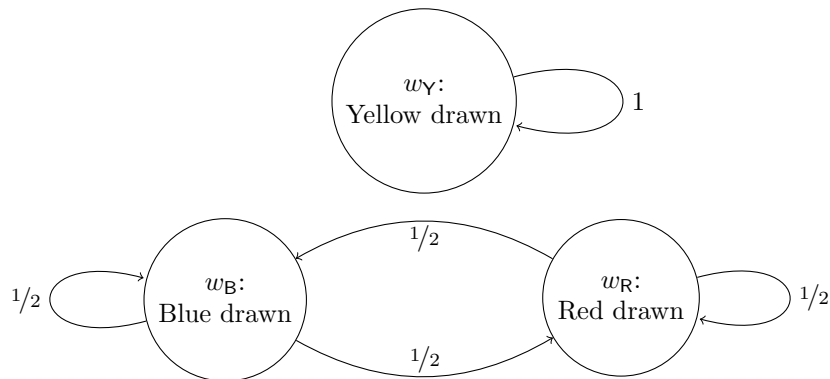


Figure 1: Example of a probabilistic modal structure.

In this example the space is finite so we can represent the measure by degree of accessibility relations. In this depiction we have not represented the arrows which would be labelled by 0. The base language for this example will include propositional variables Y, B and R representing which colour ball is drawn.

At the first stage B is evaluated positively in w_B and negatively in the other worlds. So using the frame we see that at the second stage we should now evaluate $P = (\ulcorner B \urcorner, \ulcorner 1/2 \urcorner)$ positively in w_B and w_R and negatively in w_Y .

⁶This restriction allows us to have a name for each member of the domain and therefore makes the presentation easier since we can then give the semantics without mentioning open formulas and variable assignments.

To formalise the evaluation procedure we need to record what has been evaluated positively at each world. We formally do this by using an evaluation function giving the codes of the sentences which are evaluated positively at each world.

Definition 2.5. An *evaluation function*, f , assigns to each world, w , a set $f(w) \subseteq \mathbb{N}$.

We can now proceed to give a formal analysis of the evaluation procedure. We do this by developing a definition of $\Theta(f)$ which is the evaluation function given by another step of reasoning. So if f gives the sentences that we have so far evaluated positively then $\Theta(f)$ gives the sentences that one can evaluate positively at the next stage. This can formally be understood as a satisfaction relation $(w, f) \models_{\mathcal{M}}^{SKP}$ given by $(w, f) \models_{\mathcal{M}}^{SKP} \varphi \iff \varphi \in \Theta(f)$.

At the zero-th stage one generally starts without having evaluated any sentence either way. This can be given by an evaluation function f_0 with $f_0(w) = \emptyset$ for all w .

For a sentence which does not involve truth or probability it can be evaluated positively or negatively just by considering $\mathbb{M}(w)$. So we define:

- For φ a sentence of L , $\varphi \in \Theta(f)(w) \iff \mathbb{M}(w) \models \varphi$
- For φ a sentence of L , $\neg\varphi \in \Theta(f)(w) \iff \mathbb{M}(w) \not\models \varphi$

This will give the correct evaluations to the sentences of L , for example $0 = 0 \in \Theta(f_0)(w)$ and $\neg 0 = 1 \in \Theta(f_0)(w)$.

To evaluate a sentence $\top\varphi^\top$ we first evaluate φ . If φ was evaluated positively then we can now evaluate $\top\varphi^\top$ positively and similarly for if it was evaluated negatively. However if φ was not evaluated as either then we still do not evaluate $\top\varphi^\top$ either way. This is described by the clauses:

- $\top\varphi^\top \in \Theta(f) \iff \#\varphi \in f(w)$
- $\neg\top\varphi^\top \in \Theta(f) \iff \#\neg\varphi \in f(w)$

For example we get that $\top 0 = 0^\top \in \Theta(\Theta(f_0))(w)$ and $\neg\top 0 = 1^\top \in \Theta(\Theta(f_0))(w)$.

To describe the cases for probability we consider the fragment of a probabilistic modal frame which is pictured in Fig. 2. We consider how one should evaluate $P_{\geq}(\top\varphi^\top, \top\alpha^\top)$ for different values of α .

$P_{\geq}(\top\varphi^\top, \top 0.3^\top)$ will be evaluated positively by $\Theta(f)$ because the measure of the worlds where φ is evaluated positively is $1/3 = 0.3\dot{3}$ which is larger than 0.3. $P_{\geq}(\top\varphi^\top, \top 0.7^\top)$ will be evaluated negatively by $\Theta(f)$ because however φ gets evaluated in w_3 there will be too many worlds where φ is already evaluated negatively for the measure of the worlds where it is evaluated positively to become larger than 0.7. We evaluate $P_{\geq}(\top\varphi^\top, \top 0.5^\top)$ neither way because if φ was to become evaluated in w_3 the measure of the worlds where φ is evaluated positively would become either $0.3\dot{3}$ or $0.6\dot{6}$ so we need to retain the flexibility that $P_{\geq}(\top\varphi^\top, \top 0.5^\top)$ can later be evaluated either positively or negatively depending on how φ is evaluated at w_3 .

We therefore give the definition

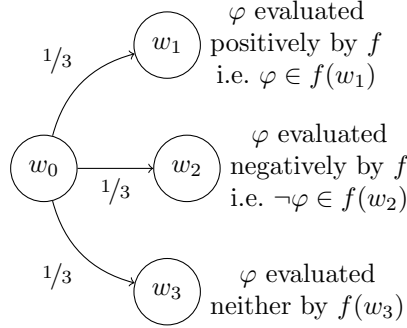


Figure 2: A fragment of a probabilistic modal structure representing the information required to evaluate $P_{\geq}(\ulcorner \varphi \urcorner, \ulcorner \alpha \urcorner)$ in $\Theta(f)(w_0)$.

- $P_{\geq}(\ulcorner \varphi \urcorner, \ulcorner \alpha \urcorner) \in \Theta(f)(w) \iff m_w\{v \mid \# \varphi \in f(v)\} \geq \alpha$
- $\neg P_{\geq}(\ulcorner \varphi \urcorner, \ulcorner \alpha \urcorner) \in \Theta(f)(w) \iff m_w\{v \mid \# \neg \varphi \in f(v)\} > 1 - \alpha$

$m_w\{v \mid \# \neg \varphi \in f(v)\} > 1 - \alpha$ exactly captures the requirement that however φ becomes evaluated in the worlds where it is not currently evaluated the measure with this new evaluation function will still not be $\geq \alpha$, i.e. for all g consistent⁷, extending f , $m_w\{v \mid \# \varphi \in g(v)\} \not\geq \alpha$.⁸

In this example we saw that the probability of φ is given by a range. This is described pictorially in Fig. 3

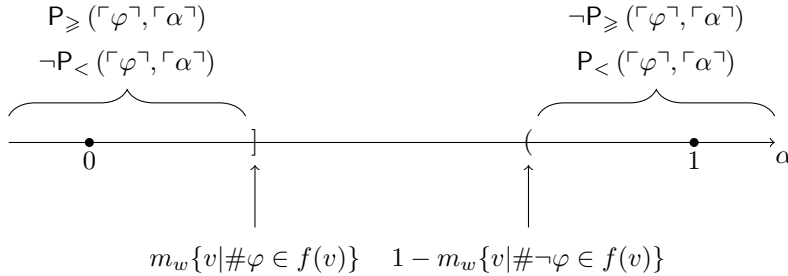


Figure 3: How $f(w)$ evaluates the probability of φ .

We lastly need to give the definitions for the connectives and quantifiers. For example we need to say how $\varphi \vee \neg \varphi$ should be evaluated if φ is itself evaluated neither way. For this we directly use the definitions in strong Kleene three valued

⁷We say g is consistent if $\varphi \in g(w) \implies \neg \varphi \notin g(w)$

⁸In this paper we are only considering consistent evaluation functions, but this definition will also apply when we want to work with non-consistent evaluation functions where one may then have that $P_{\geq}(\ulcorner \varphi \urcorner, \ulcorner \alpha \urcorner) \in \Theta(f)(w)$ and $\neg P_{\geq}(\ulcorner \varphi \urcorner, \ulcorner \alpha \urcorner) \in \Theta(f)(w)$.

logic which was the scheme that Kripke focused on and there has been much work following him in this. For example this has that $\varphi \vee \psi \in \Theta(f)(w) \iff \varphi \in \Theta(f)(w)$ or $\psi \in \Theta(f)(w)$, so if φ is evaluated neither way then so will $\varphi \vee \neg\varphi$. The advantage of this scheme is that it is defined recursively so it works well with an axiomatic approach.

This fully defines $\Theta(f)$. We only used the question of whether φ can now be evaluated positively, i.e. if $\varphi \in \Theta(f)$, as motivating the definition. We formally understand it as a definition of a three valued semantics $(w, f) \models_{\mathcal{M}}^{SKP}$ and we will later define $(w, f) \models_{\mathcal{M}}^{SKP} \varphi \iff \varphi \in \Theta(f)$. We sum up our discussion in the formal definition of $(w, f) \models_{\mathcal{M}}^{SKP}$.

Definition 2.6. For \mathcal{M} a probabilistic modal structure, $w \in W$ and f an evaluation function, define $(w, f) \models_{\mathcal{M}}^{SKP}$ by positive induction on the complexity of the formula as follows.

- $(w, f) \models_{\mathcal{M}}^{SKP} \varphi \iff \mathbb{M}(w) \models \varphi$ for φ a sentence of L
- $(w, f) \models_{\mathcal{M}}^{SKP} \neg\varphi \iff \mathbb{M}(w) \not\models \varphi$ for φ a sentence of L
- $(w, f) \models_{\mathcal{M}}^{SKP} \top t \iff t^{\mathbb{N}} \in f(w)$ and $t^{\mathbb{N}} \in \text{Sent}_{L_P, T}$
- $(w, f) \models_{\mathcal{M}}^{SKP} \neg\top t \iff \neg t^{\mathbb{N}} \in f(w)$ or $t^{\mathbb{N}} \notin \text{Sent}_{L_P, T}$
- $(w, f) \models_{\mathcal{M}}^{SKP} P_{\geq}(t, s) \iff m_w\{v | t^{\mathbb{N}} \in f(v)\} \geq \text{rat}(s^{\mathbb{N}})$ and $s^{\mathbb{N}} \in \text{Rat}$
- $(w, f) \models_{\mathcal{M}}^{SKP} \neg P_{\geq}(t, s) \iff m_w\{v | \neg t^{\mathbb{N}} \in f(v)\} > 1 - \text{rat}(s^{\mathbb{N}})$ or $s^{\mathbb{N}} \notin \text{Rat}$
- $(w, f) \models_{\mathcal{M}}^{SKP} \neg\neg\varphi \iff (w, f) \models_{\mathcal{M}}^{SKP} \varphi$
- $(w, f) \models_{\mathcal{M}}^{SKP} \varphi \vee \psi \iff (w, f) \models_{\mathcal{M}}^{SKP} \varphi$ or $(w, f) \models_{\mathcal{M}}^{SKP} \psi$
- $(w, f) \models_{\mathcal{M}}^{SKP} \neg(\varphi \vee \psi) \iff (w, f) \models_{\mathcal{M}}^{SKP} \neg\varphi$ and $(w, f) \models_{\mathcal{M}}^{SKP} \neg\psi$
- $(w, f) \models_{\mathcal{M}}^{SKP} \exists x\varphi(x) \iff (w, f) \models_{\mathcal{M}}^{SKP} \varphi[\bar{n}/x]$ for some $n \in \mathbb{N}$.
- $(w, f) \models_{\mathcal{M}}^{SKP} \neg\exists x\varphi(x) \iff (w, f) \models_{\mathcal{M}}^{SKP} \neg\varphi[\bar{n}/x]$ for all $n \in \mathbb{N}$

Note that we have omitted the connectives $\wedge, \rightarrow, \leftrightarrow, \forall$. We shall take them as defined connectives. For example $\varphi \wedge \psi := \neg(\neg\varphi \vee \neg\psi)$.

The only difference to the standard definition is the addition of the clauses for probability.

Now we give the definition of Θ in terms of $(w, f) \models_{\mathcal{M}}^{SKP}$.

Definition 2.7. Define Θ a function from evaluation functions to evaluation functions by

$$\Theta(f)(w) := \{\#\varphi | (w, f) \models_{\mathcal{M}}^{SKP} \varphi\}$$

We now consider an example of how this works for the “unproblematic” sentences.

Consider again the example in Fig. 1. Take any f . Observe that

$$(w_B, f) \models_{\mathcal{M}}^{SKP} \text{Blue} \text{ and } (w_R, f) \models_{\mathcal{M}}^{SKP} \neg \text{Blue}$$

so

$$\# \text{Blue} \in \Theta(f)(w_B) \text{ and } \# \neg \text{Blue} \in \Theta(f)(w_R).$$

Therefore

$$(w_B, \Theta(f)) \models_{\mathcal{M}}^{SKP} P_=(\ulcorner \text{Blue} \urcorner, \ulcorner 1/2 \urcorner) \text{ and similarly for } w_R$$

so

$$\# P_=(\ulcorner \text{Blue} \urcorner, \ulcorner 1/2 \urcorner) \in \Theta(\Theta(f))(w_B) \text{ and similarly for } w_R.$$

Then by similar reasoning

$$(w_B, \Theta(\Theta(f))) \models_{\mathcal{M}}^{SKP} P_=(P_=(\ulcorner \text{Blue} \urcorner, \ulcorner 1/2 \urcorner), \ulcorner 1 \urcorner)$$

so

$$\# P_=(P_=(\ulcorner \text{Blue} \urcorner, \ulcorner 1/2 \urcorner), \ulcorner 1 \urcorner) \in \Theta(\Theta(\Theta(f)))(w_B).$$

These sentences have an easy translation into the operator language and such sentences will be given point-valued probabilities and be evaluated positively or negatively by some $\Theta^n(f)$.

Θ can only give truth values to sentences which were previously undefined, it cannot *change* the truth values of sentences. This idea can be formalised by saying that Θ is monotone.

Lemma 2.8 (Θ is monotone). *If for all w $f(w) \subseteq g(w)$, then also for all w $\Theta(f)(w) \subseteq \Theta(g)(w)$.*

Proof. Take some evaluation functions f and g with $f(w) \subseteq g(w)$ for all w . It suffices to prove that if $(w, f) \models \varphi$ then $(w, g) \models \varphi$. We do this by induction on the positive complexity of φ . \square

This fact ensures that there are fixed points of the operator Θ , i.e. evaluation functions f with $f = \Theta(f)$. These are evaluation functions where the process of evaluation doesn't lead to any new information.

Corollary 2.9 (Θ has fixed points). *For every \mathcal{M} there is some consistent evaluation function f such that $\Theta(f) = f$. We call evaluation functions with $f = \Theta(f)$ fixed point evaluation functions.*

If φ is grounded in facts that are not about truth or probability then this process of evaluation will terminate in such facts and the sentence will be evaluated appropriately in a fixed point of Θ . They will also therefore be given a point-valued probability as is desired. This will cover sentences which are expressible in the operator language and therefore shows that this semantics extends an operator semantics, a minimal adequacy requirement for any proposed semantics.⁹

⁹We can show for the natural translation function ρ from the operator language as presented in (Heifetz & Mongin, 2001) to the predicate language and f a fixed point one has

$$w \models_{\mathcal{M}} \varphi \iff (w, f) \models_{\mathcal{M}}^{SKP} \rho(\varphi)$$

However we will get more, for example $0 = 0 \vee \lambda$ will be evaluated positively in each world and so be assigned probability 1, i.e. $P_{=}(\ulcorner 0 = 0 \vee \lambda \urcorner, \ulcorner 1 \urcorner)$ will also be evaluated positively in a fixed point.

The fixed points have some nice properties.

Proposition 2.10. *For f a fixed point of Θ we have*

$$\varphi \in f(w) \iff (w, f) \models_{\mathcal{M}}^{SKP} \varphi$$

Therefore we have

- $(w, f) \models_{\mathcal{M}}^{SKP} \top \ulcorner \varphi \urcorner \iff (w, f) \models_{\mathcal{M}}^{SKP} \varphi$
- $(w, f) \models_{\mathcal{M}}^{SKP} P_{\geq}(\ulcorner \varphi \urcorner, \ulcorner \alpha \urcorner) \iff m_w\{v \mid (v, f) \models_{\mathcal{M}}^{SKP} \varphi\} \geq \alpha$
- $(w, f) \models_{\mathcal{M}}^{SKP} P_{>}(\ulcorner \varphi \urcorner, \ulcorner \alpha \urcorner) \iff m_w\{v \mid (v, f) \models_{\mathcal{M}}^{SKP} \varphi\} > \alpha$

Proof. Follows immediately from Definitions 2.6 and 2.7 □

These are exactly the clauses that we wished to satisfy in a classical semantics but we saw was not possible because of liar-like paradoxes. This shows that we can satisfy these if we instead consider partial models.

2.3 The Classical Semantics

We used three-valued logic to formalise the evaluation procedure but along with Kripke we intend the third truth value to be not a real truth value but just as “not yet defined”. We do not wish our final proposal to be given in three-valued logic so we shall “close-off” the fixed points. We will define *the induced model given by \mathcal{M} and f at w , $\text{IndMod}_{\mathcal{M}}[f, w]$* , by putting the unevaluated facts about truth and probability to false. This is described pictorially by altering Fig. 3 to Fig. 4.

$\text{IndMod}_{\mathcal{M}}[f, w] \models$:

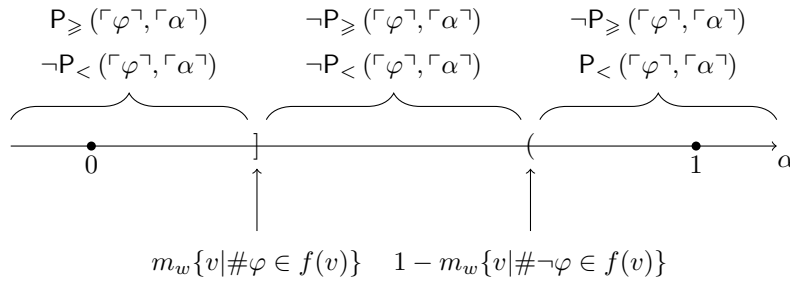


Figure 4: How $\text{IndMod}_{\mathcal{M}}[f, w]$ evaluates the probability of φ

It is defined formally as follows.

Definition 2.11. Define $\text{IndMod}_{\mathcal{M}}[f, w]$ to be a (classical) first order model in the language $L_{P, \top}$ which has the domain \mathbb{N} , interprets the predicates from L as is specified by $\mathbb{M}(w)$, and interprets the other predicates by:

- $\text{IndMod}_{\mathcal{M}}[f, w] \models \top \bar{n} \iff (w, f) \models_{\mathcal{M}}^{SKP} \top \bar{n}$
- $\text{IndMod}_{\mathcal{M}}[f, w] \models P_{\geq}(\bar{n}, \bar{m}) \iff (w, f) \models_{\mathcal{M}}^{SKP} P_{\geq}(\bar{n}, \bar{m})$

This will satisfy.

Proposition 2.12. For \mathcal{M} a probabilistic modal structure, f an evaluation function and $w \in W$,

- $\text{IndMod}_{\mathcal{M}}[f, w] \models P_{>}(\ulcorner \varphi \urcorner, \ulcorner \alpha \urcorner) \iff (w, f) \models_{\mathcal{M}}^{SKP} P_{>}(\ulcorner \varphi \urcorner, \ulcorner \alpha \urcorner)$
- $\text{IndMod}_{\mathcal{M}}[f, w] \models P_{\leq}(\ulcorner \varphi \urcorner, \ulcorner \alpha \urcorner) \iff (w, f) \models_{\mathcal{M}}^{SKP} P_{\leq}(\ulcorner \varphi \urcorner, \ulcorner \alpha \urcorner)$
- $\text{IndMod}_{\mathcal{M}}[f, w] \models P_{<}(\ulcorner \varphi \urcorner, \ulcorner \alpha \urcorner) \iff (w, f) \models_{\mathcal{M}}^{SKP} P_{<}(\ulcorner \varphi \urcorner, \ulcorner \alpha \urcorner)$
- $\text{IndMod}_{\mathcal{M}}[f, w] \models P_{=}(\ulcorner \varphi \urcorner, \ulcorner \alpha \urcorner) \iff (w, f) \models_{\mathcal{M}}^{SKP} P_{=}(\ulcorner \varphi \urcorner, \ulcorner \alpha \urcorner)$

These models are classical models of the language $L_{P, \top}$, but the “inner logic of \top ”¹⁰ will not be classical and not every sentence will be assigned a point valued probability. For example we will have $\neg \top \ulcorner \lambda \urcorner$, $\neg \top \neg \ulcorner \lambda \urcorner$, $\neg P_{>}(\ulcorner \lambda \urcorner, \ulcorner 0 \urcorner)$ and $\neg P_{<}(\ulcorner \lambda \urcorner, \ulcorner 1 \urcorner)$.

These models, for fixed points f , are our proposal for the semantics of the language.

3 Observations and Comments on the Semantics

3.1 Introspection

We were interested in introspection since it is inconsistent with probabilism in languages which allow for self-referential probabilities and was discussed by (Caie, 2013) and (Christiano et al., n.d.).

A probabilistic modal structure which has the property that

$$\text{For all } w, m_w\{v \mid m_v = m_w\} = 1$$

will satisfy introspection in the operator language. That is

$$\begin{aligned} P_{\geq \alpha}(\varphi) &\rightarrow P_{=1}(P_{\geq \alpha}(\varphi)) \\ \neg P_{\geq \alpha}(\varphi) &\rightarrow P_{=1}(\neg P_{\geq \alpha}(\varphi)) \end{aligned}$$

Such probabilistic modal structures will also satisfy introspection in the predicate setting if the principles are expressed using a truth predicate.

¹⁰By this we mean the logical laws that hold in the inside applications of the truth predicate.

Proposition 3.1. *Let \mathcal{M} be such that $m_w\{v|m_v = m_w\} = 1$ for all w . Then for any evaluation function f and world w ,*

- *If $(w, f) \models_{\mathcal{M}}^{SKP} P_{\geq}(\ulcorner\varphi\urcorner, \ulcorner\alpha\urcorner)$ then $(w, f) \models_{\mathcal{M}}^{SKP} P_{\geq}(\ulcorner P_{\geq}(\ulcorner\varphi\urcorner, \ulcorner\alpha\urcorner)\urcorner, \ulcorner 1\urcorner)$*
- *If $(w, f) \models_{\mathcal{M}}^{SKP} \neg P_{\geq}(\ulcorner\varphi\urcorner, \ulcorner\alpha\urcorner)$ then $(w, f) \models_{\mathcal{M}}^{SKP} P_{\geq}(\ulcorner \neg P_{\geq}(\ulcorner\varphi\urcorner, \ulcorner\alpha\urcorner)\urcorner, \ulcorner 1\urcorner)$*

And similarly for $P_{>}$ etc.

By the definition of $\text{IndMod}_{\mathcal{M}}[f, w]$ we therefore have ¹¹

- $\text{IndMod}_{\mathcal{M}}[\Theta(\Theta(f)), w] \models \text{T}P_{\geq}(\ulcorner\varphi\urcorner, \ulcorner\alpha\urcorner)\urcorner \rightarrow P_{\geq}(\ulcorner P_{\geq}(\ulcorner\varphi\urcorner, \ulcorner\alpha\urcorner)\urcorner, \ulcorner 1\urcorner)$
- $\text{IndMod}_{\mathcal{M}}[\Theta(\Theta(f)), w] \models \text{T}\neg P_{\geq}(\ulcorner\varphi\urcorner, \ulcorner\alpha\urcorner)\urcorner \rightarrow P_{\geq}(\ulcorner \neg P_{\geq}(\ulcorner\varphi\urcorner, \ulcorner\alpha\urcorner)\urcorner, \ulcorner 1\urcorner)$

And similarly for $P_{>}$ etc.

Therefore for f a fixed point evaluation function,

- $\text{IndMod}_{\mathcal{M}}[f, w] \models \text{T}P_{\geq}(\ulcorner\varphi\urcorner, \ulcorner\alpha\urcorner)\urcorner \rightarrow P_{\geq}(\ulcorner P_{\geq}(\ulcorner\varphi\urcorner, \ulcorner\alpha\urcorner)\urcorner, \ulcorner 1\urcorner)$
- $\text{IndMod}_{\mathcal{M}}[f, w] \models \text{T}\neg P_{\geq}(\ulcorner\varphi\urcorner, \ulcorner\alpha\urcorner)\urcorner \rightarrow P_{\geq}(\ulcorner \neg P_{\geq}(\ulcorner\varphi\urcorner, \ulcorner\alpha\urcorner)\urcorner, \ulcorner 1\urcorner)$

And similarly for $P_{>}$ etc.¹²

One might give an explanation for this as follows. To answer the question of whether

$$\text{IndMod}_{\mathcal{M}}[\Theta(\Theta(f)), w] \models \text{T}P_{\geq}(\ulcorner\varphi\urcorner, \ulcorner\alpha\urcorner)\urcorner \rightarrow P_{\geq}(\ulcorner P_{\geq}(\ulcorner\varphi\urcorner, \ulcorner\alpha\urcorner)\urcorner, \ulcorner 1\urcorner)$$

one needs only to answer the question of whether

$$\text{IndMod}_{\mathcal{M}}[f, v] \models \varphi$$

and use the definitions. However, to answer the question of whether

$$\text{IndMod}_{\mathcal{M}}[\Theta(\Theta(f)), w] \models P_{\geq}(\ulcorner\varphi\urcorner, \ulcorner\alpha\urcorner) \rightarrow P_{\geq}(\ulcorner P_{\geq}(\ulcorner\varphi\urcorner, \ulcorner\alpha\urcorner)\urcorner, \ulcorner 1\urcorner)$$

one needs to answer both the questions of

$$\text{IndMod}_{\mathcal{M}}[\Theta(f), v] \models \varphi$$

and

$$\text{IndMod}_{\mathcal{M}}[f, v] \models \varphi$$

¹¹In fact the quantified versions

- $\text{IndMod}_{\mathcal{M}}[\Theta(\Theta(f)), w] \models \forall a \forall x (\text{TP}_{\geq}(x, a) \rightarrow P_{\geq}(P_{\geq}(x, a), 1))$
- $\text{IndMod}_{\mathcal{M}}[\Theta(\Theta(f)), w] \models \forall a \forall x (\text{T}\neg P_{\geq}(x, a) \rightarrow P_{\geq}(\neg P_{\geq}(x, a), 1))$

are satisfied but we do not present this because it is not important for our point and we don't feel that it is worth yet introducing this general \triangleright notation which is introduced in ??

¹²We could equivalently formalise the principles as

- $P_{\geq}(\ulcorner \text{T}\ulcorner\varphi\urcorner, \ulcorner\alpha\urcorner\urcorner \rightarrow P_{\geq}(\ulcorner P_{\geq}(\ulcorner\varphi\urcorner, \ulcorner\alpha\urcorner)\urcorner, \ulcorner 1\urcorner)$
- $\neg P_{\geq}(\ulcorner \text{T}\ulcorner\varphi\urcorner, \ulcorner\alpha\urcorner\urcorner \rightarrow P_{\geq}(\ulcorner \neg P_{\geq}(\ulcorner\varphi\urcorner, \ulcorner\alpha\urcorner)\urcorner, \ulcorner 1\urcorner)$

and use the definitions.

This shows that when one asks for the version with the T predicate one only asks about properties of the probabilistic modal structure and not about how Θ works. This is therefore a well-motivated way to express the principle in this framework.

This is an example of a more general strategy which one might employ in formulating desiderata such as introspection. The strategy comes from (Stern, 2014a) where he suggests that following the strategy of “avoiding introduction and elimination of the modal predicate independently of the truth predicate” might allow one to avoid paradoxes. Moreover he says:

“[This strategy] seems to be well motivated if one adopts the deflationist idea that quotation and disquotation are the function of the truth predicate. Consequently, quotation and disquotation of sentences is not the task of the modal predicate and in formulating modal principles we should therefore avoid the introduction or elimination of the modal predicates without the detour via the truth predicate.”

So if one accepts this as an appropriate formulation of introspection we have that introspection and probabilism are compatible.¹³

Further work should be done to see how one should state other principles and whether expressing them allows one to avoid paradoxical contradictions arising only from the self-referential nature of the language.

We next show a nice feature of the construction, namely that it can account for the Gaifman condition.

3.2 Gaifman Condition

A function $F : \text{Sent}_L \rightarrow \mathbb{R}$ is said to satisfy the *Gaifman condition* if

$$\text{For all } \varphi, F(\exists x\varphi(x)) = \lim_{n \rightarrow \infty} F\left(\bigvee_{0 \leq i \leq n} \varphi(\bar{i})\right)$$

This in part captures the idea that the domain is exactly $0, 1, 2, \dots$. This was called σ -additivity in (Leitgeb, 2008) and (Leitgeb, 2012).

As was mentioned in the introduction both (Leitgeb, 2012) and (Christiano et al., n.d.) face a challenge from the Gaifman condition because both Christiano’s requirements and the ω -stage of Leitgeb’s construction¹⁴ lead to a for-

¹³In fact in this setting the principles $P_{\geq}(\ulcorner \varphi \urcorner, \ulcorner \alpha \urcorner) \rightarrow P_{=}(\ulcorner P_{\geq}(\ulcorner \varphi \urcorner, \ulcorner \alpha \urcorner) \urcorner, \ulcorner 1 \urcorner)$ and $P_{<}(\ulcorner \varphi \urcorner, \ulcorner \alpha \urcorner) \rightarrow P_{=}(\ulcorner P_{<}(\ulcorner \varphi \urcorner, \ulcorner \alpha \urcorner) \urcorner, \ulcorner 1 \urcorner)$ are consistent even though $P_{\geq}(\ulcorner \varphi \urcorner, \ulcorner \alpha \urcorner) \rightarrow P_{=}(\ulcorner P_{\geq}(\ulcorner \varphi \urcorner, \ulcorner \alpha \urcorner) \urcorner, \ulcorner 1 \urcorner)$ and $\neg P_{\geq}(\ulcorner \varphi \urcorner, \ulcorner \alpha \urcorner) \rightarrow P_{=}(\ulcorner \neg P_{\geq}(\ulcorner \varphi \urcorner, \ulcorner \alpha \urcorner) \urcorner, \ulcorner 1 \urcorner)$ are not. This is because for the problematic sentences in frames where $m_w\{v|m_v = m_w\}$ will not satisfy $P_{<}(\ulcorner \varphi \urcorner, \ulcorner \alpha \urcorner)$ or $P_{\geq}(\ulcorner \varphi \urcorner, \ulcorner \alpha \urcorner)$. We therefore see that this is an alternative way of showing that introspection can remain consistent in this setting.

¹⁴Which is given by the axiomatic theory PT_2 in his paper

mula $\varphi(x)$ such that for each n $\mathbf{P}^\Gamma \varphi(\bar{0}) \wedge \dots \wedge \varphi(\bar{n})^\neg = 1$ but $\mathbf{P}^\Gamma \forall x \varphi(x)^\neg = 0$.¹⁵

Our theory does not have this flaw. However since our sentences are sometimes given ranges of probabilities instead of points we should reformulate the definition of the Gaifman condition to apply to ranges. Let $\underline{\mathbf{P}}$ and $\bar{\mathbf{P}}$ denote the upper and lower bounds of the range of probabilities assigned to φ . More carefully:

Definition 3.2. Fix some probabilistic modal structure \mathcal{M} , evaluation function f and world w . Define

$$\begin{aligned}\underline{\mathbf{P}}(\varphi) &:= \sup\{\alpha \mid \text{IndMod}_{\mathcal{M}}[f, w] \models \mathbf{P}_{\geq}(\ulcorner \varphi^\neg, \urcorner \alpha^\neg)\} \\ \bar{\mathbf{P}}(\varphi) &:= \inf\{\alpha \mid \text{IndMod}_{\mathcal{M}}[f, w] \models \mathbf{P}_{<}(\ulcorner \varphi^\neg, \urcorner \alpha^\neg)\}\end{aligned}$$

This can be seen as in Fig. 5

$\text{IndMod}_{\mathcal{M}}[f, w] \models$:

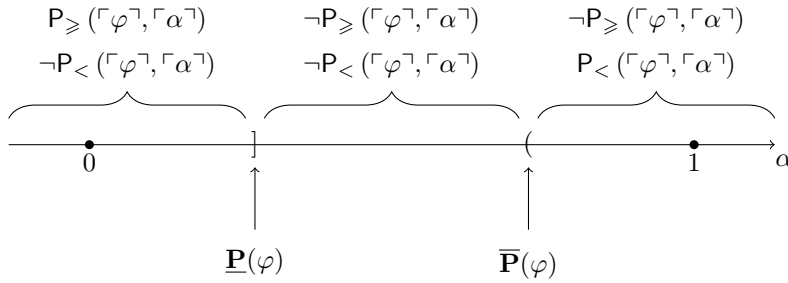


Figure 5: Definition of $\underline{\mathbf{P}}(\varphi)$ and $\bar{\mathbf{P}}(\varphi)$

Definition 3.3. We can say that \mathbf{P} as given by $\text{IndMod}_{\mathcal{M}}[f, w]$ satisfies the *extended Gaifman condition* if $\underline{\mathbf{P}}(\forall x \varphi(x)) = \lim_{n \rightarrow \infty} \underline{\mathbf{P}}(\varphi(\bar{0}) \wedge \dots \wedge \varphi(\bar{n}))$ and similarly for $\bar{\mathbf{P}}$.

If we consider probabilistic modal structures where all the measures m_w are σ -additive then the extended Gaifman condition will be satisfied.

Theorem 3.4. *If \mathcal{M} is such that every m_w is σ -additive on an algebra of subsets containing the sets of the form $\{v \mid n \in f(v)\}$ and f is a fixed point, then \mathbf{P} as given by $\text{IndMod}_{\mathcal{M}}[f, w]$ will satisfy the extended Gaifman condition.*

This is the form of the Gaifman condition which is appropriate in the context where we are dealing with interval-valued probabilities and therefore shows that our approach is not restricted to finite additivity.

¹⁵For Leitgeb this is given by $\varphi(x) := \overbrace{\mathbf{P}^\Gamma \mathbf{P}^\Gamma \dots \mathbf{P}^\Gamma \delta^\neg = 1 \dots^\neg = 1^\neg = 1}^x$ where δ is the McGee sentence which has the property $\text{PA}^{L_P, \top} \vdash \neg \delta \leftrightarrow \forall n \overbrace{\mathbf{P}^\Gamma \mathbf{P}^\Gamma \dots \mathbf{P}^\Gamma \delta^\neg = 1 \dots^\neg = 1^\neg = 1}^n$. For Christiano et al. this is given by $\varphi(x) := 1 - 1/x < \mathbf{P}^\Gamma \epsilon^\neg < 1 + 1/x$ where ϵ has the property $\text{PA}^{L_P, \top} \vdash \epsilon \leftrightarrow \mathbf{P}^\Gamma \epsilon^\neg < 1$.

3.3 \mathbf{P} is an SK-probability

Although the models $\text{IndMod}_{\mathcal{M}}[f, w]$ are classical they loose aspects of traditional probability functions. $\text{IndMod}_{\mathcal{M}}[f, w]$ does not assign particular values for the probability of φ but can be seen instead as providing ranges. As such $\text{IndMod}_{\mathcal{M}}[f, w]$ provides us with two functions to consider, these are $\underline{\mathbf{P}}$ and $\overline{\mathbf{P}}$ as given in Definition 3.2.

Both these functions loose nice properties which one would expect from classical probabilities, for example $\underline{\mathbf{P}}(\lambda \vee \neg\lambda) = 0$, and $\overline{\mathbf{P}}(\lambda \wedge \neg\lambda) = 1$. However we can show that $\underline{\mathbf{P}}$ and $\overline{\mathbf{P}}$ can be seen as non-classical probabilities in the sense of (Williams, 2014) over logics which arise from Kleene's strong three valued scheme. In particular, $\underline{\mathbf{P}}$ is a non-classical probability over Kleene logic K_3 which is defined by truth preservation in Kleene assignments and $\overline{\mathbf{P}}$ is a non-classical probability over LP -logic which is defined by falsity anti-preservation in Kleene assignments.

4 An Axiomatic System

In the last section of this paper we present an axiomatic theory for this semantic construction. This will allow us to better reason about this semantics.

To present this we need to provide some more notation.

Notation 4.1. We represent the interpretation function $^{\mathbb{N}}$ function by $^{\circ}$, but this is understood not to be a function symbol in our language. We therefore have that for any term t , $\ulcorner t \urcorner^{\circ} = \ulcorner t^{\mathbb{N}} \urcorner$ is a true, non-atomic formula of our language.

We shall use Rat , Sent_L , $\text{Sent}_{P, \top}$, Cterm_L to denote the set of codes of rational numbers, sentences of L , sentences of $L_{P, \top}$, and closed terms of L respectively. We shall also assume that we have these as predicates in the language. If \triangleright is a syntactic operation we shall assume we have a function symbol \triangleright in our language representing it¹⁶. Exceptions are the substitution function which we represent by $x(y/v)$, and $^{\circ}$ which we already introduced. We shall similarly represent operations on the rationals, for example we have a function symbol \dagger . We shall still use $<$ to represent the ordering on the rational numbers, so $\# \alpha < \# \beta \iff \alpha < \beta$. Finally we use $\forall \tau \varphi(\tau)$ or $\forall \rho \varphi(\rho)$ as shorthand for $\forall x(\text{Cterm}_L(x) \rightarrow \varphi(x))$.

We now present the axiomatic system.

Definition 4.2. Remember we introduced the following abbreviations:

- $\mathbf{P}_{>}(s, t) := \exists a \succ t(\mathbf{P}_{\geq}(s, a))$
- $\mathbf{P}_{\leq}(s, t) := \mathbf{P}_{\geq}(\neg s, 1 - t)$
- $\mathbf{P}_{<}(s, t) := \mathbf{P}_{>}(\neg s, 1 - t)$

¹⁶Observe that these are not contingent function symbols so this ensures that $t^{\mathbb{N}}$ is still well-defined

- $P_=(s, t) := P_{\geq}(s, t) \wedge P_{\leq}(s, t)$

Define ProbKF to be given by the following axioms, added to an axiomatisation of classical logic.

- KF, the axioms for truth:
 1. $PA^{L_{P, \top}}$ the axioms of Peano Arithmetic with the induction schema extended to $L_{P, \top}$.
 2. $\forall \tau \forall \rho (\top \rho \equiv \tau \leftrightarrow \rho^\circ = \tau^\circ)$
 3. $\forall \tau \forall \rho (\top \neg \rho \equiv \tau \leftrightarrow \neg \rho^\circ = \tau^\circ)$
 4. $\forall \tau_1 \dots \tau_n (\top Q \tau_1 \dots \tau_n \leftrightarrow Q \tau_1^\circ \dots \tau_n^\circ)$ for each n -ary predicate Q of L
 5. $\forall \tau_1 \dots \tau_n (\top \neg Q \tau_1 \dots \tau_n \leftrightarrow \neg Q \tau_1^\circ \dots \tau_n^\circ)$ for each n -ary predicate Q of L
 6. $\forall x (\text{Sent}_{P, \top}(x) \rightarrow (\top \neg \neg x \leftrightarrow \top x))$
 7. $\forall x (\text{Sent}_{P, \top}(x \wedge y) \rightarrow (\top x \wedge y \leftrightarrow \top x \wedge \top y))$
 8. $\forall x (\text{Sent}_{P, \top}(x \wedge \neg y) \rightarrow (\top \neg x \wedge y \leftrightarrow \top \neg x \vee \top \neg y))$
 9. $\forall x (\text{Sent}_{P, \top}(\forall v x) \rightarrow (\top \forall v x \leftrightarrow \forall y \top(x(y/v))))$
 10. $\forall x (\text{Sent}_{P, \top}(\forall v x) \rightarrow (\top \neg \forall v x \leftrightarrow \exists y \top(\neg x(y/v))))$
 11. $\forall \tau (\top \top \tau \leftrightarrow \top \tau^\circ)$
 12. $\forall \tau (\top \neg \top \tau \leftrightarrow (\top \neg \tau^\circ \vee \neg \text{Sent}_{P, \top}(\tau^\circ)))$
 13. $\forall x (\top x \rightarrow \text{Sent}_{P, \top}(x))$
- InteractionAx, the axioms for the interaction of truth and probability:¹⁷
 14. $\forall \tau \forall \rho (\top P_{\geq}(\tau, \rho) \leftrightarrow P_{\geq}(\tau^\circ, \rho^\circ))$
 15. $\forall \tau \forall \rho (\top \neg P_{\geq}(\tau, \rho) \leftrightarrow P_{<}(\tau^\circ, \rho^\circ) \vee \neg \text{Rat}(\rho^\circ))$
- The axioms which give basic facts about P_{\geq} :
 16. $\forall a (\exists x P_{\geq}(x, a) \rightarrow \text{Rat} a)$
 17. $\forall x (P_{>}(x, 0) \rightarrow \text{Sent}_{P, \top} x)$
 18. $\forall x \forall a (P_{\geq}(x, a) \rightarrow \forall b \prec a P_{\geq}(x, b))$
 19. $\forall x \forall b \in \text{Rat} ((\forall a \prec b P_{\geq}(x, a)) \rightarrow P_{\geq}(x, b))$
- Axioms which say that P acts like a probability¹⁸:
 20. $P_=(\top 0 = 0^\top, \top 1^\top)$
 21. $P_=(\top 0 = 1^\top, \top 0^\top)$

¹⁷These should be seen as the appropriate way of extending KF to the language $L_{P, \top}$, but we include them separately to highlight them.

¹⁸We use the axioms for 2-additive choquet capacities because our underlying structure might be a lattice not a boolean algebra.

22. $\forall x \forall y (\text{Sent}_{\mathbb{P}, \top}(x) \wedge \text{Sent}_{\mathbb{P}, \top}(y) \rightarrow$
 $\left(\forall a \in \text{Rat} \left((\forall b \forall c (\mathbb{P}_{\geq}(x, b) \wedge \mathbb{P}_{\geq}(y, c) \rightarrow b \dot{+} c \preceq a)) \right) \right)$
 $\left(\leftrightarrow (\forall d \forall e (\mathbb{P}_{\geq}(x \wedge y, d) \wedge \mathbb{P}_{\geq}(x \vee y, e) \rightarrow d \dot{+} e \preceq a)) \right) \right)$
23. $\frac{\top t \rightarrow \top s}{\forall a (\mathbb{P}_{\geq}(t, a) \rightarrow \mathbb{P}_{\geq}(s, a))}$

These axioms are sound, i.e. all induced models satisfy the axiomatisation. They are also complete in the presence of the ω -rule which allows one to conclude $\forall x \varphi(x)$ from all the instances of $\varphi(\bar{n})$. This is needed to fix the standard model of arithmetic which we relied on in building the semantics.

Proposition 4.3 (Soundness of ProbKF). *If $\vdash_{\text{ProbKF}} \varphi$ then for any probabilistic modal structure \mathcal{M} , fixed point evaluation function f , and $w \in W$, $\text{IndMod}_{\mathcal{M}}[f, w] \models \varphi$*

Proof. By induction on the length of the proof in ProbKF. Most of the axioms follow from Definition 2.6 using the fact that since f is a fixed point $\text{IndMod}_{\mathcal{M}}[f, w] \models \top \varphi \iff (w, f) \models_{\mathcal{M}}^{SKP} \varphi$. \square

We would additionally like to have a completeness component to the axiomatisation. To get a completeness theorem we ensure that arithmetic is fixed. To do this we add an ω -rule to the axiomatisation.

Definition 4.4. Let ProbKF^{ω} denote the system ProbKF with the added rule

$$\frac{\varphi(\bar{0}) \quad \varphi(\bar{1}) \quad \dots}{\forall x \varphi(x)}$$

for each formula $\varphi(x)$ with x as the only free variable.

Theorem 4.5 (Soundness and Completeness of ProbKF^{ω}). *$\mathcal{A} \models \text{ProbKF}^{\omega}$ if and only if there is an probabilistic structure \mathcal{M} , fixed point f and $w \in W$ such that $\mathcal{A} = \text{IndMod}_{\mathcal{M}}[f, w]$.*

This result is proved by a canonical model construction. The fact that we can produce a canonical model is independently interesting since it gives a systematic structure which one can use when doing anything with these semantics.

4.1 Proof of the Soundness and Completeness of ProbKF^{ω}

We quickly mention the soundness result before moving on to sketch a proof of the completeness component.

Theorem 4.6 (Soundness of ProbKF^{ω}). *For a probabilistic modal structure \mathcal{M} , fixed point evaluation function f and $w \in W$, we have $\text{IndMod}_{\mathcal{M}}[f, w] \models \text{ProbKF}^{\omega}$.*

Proof. Generalise the argument in Proposition 4.3 by transfinite induction on the depth of the proof. \square

We can now turn to the more interesting completeness direction.

Definition 4.7. Define a probabilistic structure, \mathcal{M} , and evaluation function f as follows:

- $W := \{w | w \text{ is a maximally ProbKF}^\omega\text{-consistent set of formulas}\}$
- Define each $\mathbb{M}(w)$ a standard model of L with $\mathbb{M}(w) \models \varphi$ iff $\varphi \in w$, for $\varphi \in \text{Sent}_L$
- $f(w) := \{n | \top \bar{n} \in w\}$
- For each $w \in W$ find $m_w : \mathcal{P}(W) \rightarrow \mathbb{R}$ probabilistic such that

$$m_w(\{v | \top \bar{n} \in v\}) := \sup\{\alpha | \mathbb{P}_{\geq}(\bar{n}, \ulcorner \alpha \urcorner) \in w\}$$

The axioms in Items 16 to 23 allow us to find such an m_w . We state this fact in the following lemma

Lemma 4.8. *For each $w \in W$ such an m_w can be found.*

Proof. We shall show that $\{[n] | n \in \mathbb{N}\}$ is a distributive lattice, where $[n] := \{v | \top \bar{n} \in f(v)\}$, and that $\mu : \{[n] | n \in \mathbb{N}\} \rightarrow \mathbb{R}$ by $\mu(\{v | \top \bar{n} \in v\}) = \sup\{\alpha | \mathbb{P}_{\geq}(\bar{n}, \ulcorner \alpha \urcorner) \in w\}$ is a monotone 2-valuation on this¹⁹. By (Zhou, 2013) we can then conclude that this can be extended to a probability function on the Boolean closure of $\{[n] | n \in \mathbb{N}\}$, and then further that it can be extended to the powerset of W . \square

We can observe that this is canonical, in the following sense.

Lemma 4.9. $\text{IndMod}_{\mathcal{M}}[f, w] \models \varphi \iff \varphi \in w$ for every $\varphi \in L_{\mathbb{P}, \top}$ and $w \in W$.²⁰

Proof. Since w is consistent in ω -logic, it has an ω -model. $\text{IndMod}_{\mathcal{M}}[f, w]$ is also a standard model by definition so we just need to check that the same properties hold of the natural numbers. We therefore need to check the cases $\varphi \in \text{Sent}$, $\varphi = \top \bar{n}$ and $\varphi = \mathbb{P}_{\geq}(\bar{n}, \bar{m})$, but these follow immediately from the definition of the canonical model. \square

The following lemma, along with Lemma 4.9 allows us to conclude that f is a fixed point, and therefore that the constructed model is in fact a probabilistic structure with a fixed point.

¹⁹I.e. that

- $\mu(W) = 1$
- $\mu(\emptyset) = 0$
- $A \subseteq B \implies \mu(A) \leq \mu(B)$
- $\mu(A) + \mu(B) = \mu(A \cap B) + \mu(A \cup B)$

²⁰In fact we only need this lemma for $\varphi \in \Pi_1^0$ since $\text{KF} \cup \text{InteractionAx}$ is a Π_1^0 theory, but we can prove it for all φ .

Lemma 4.10. *Let \mathcal{M} be a probabilistic modal structure and f be an evaluation function on \mathcal{M} . Then:*

$$f \text{ is a fixed point} \iff \forall w \in W(\text{IndMod}_{\mathcal{M}}[f, w] \models \text{KF} \cup \text{InteractionAx})$$

Proof. One direction follows from Proposition 4.3. For the other direction we work by induction on the positive complexity of φ to show that if $\text{IndMod}_{\mathcal{M}}[f, w] \models \text{KF} \cup \text{InteractionAx}$ then $\text{IndMod}_{\mathcal{M}}[f, w] \models \text{T}^{\Gamma} \varphi^{\Gamma} \iff (w, f) \models_{\mathcal{M}}^{SKP} \varphi$. \square

This lemma extends the useful result from Feferman that $(\mathbb{N}, S) \models \text{KF}$ iff S is a fixed point. This result was generalised by (Stern, 2014b) where Stern shows that KF extended by axioms for the interaction of truth with a necessity and possibility predicate analogous to Items 14 and 15 allows one to capture the fixed points. Lemma 4.10 is a minor modifications of Stern’s result.

Corollary 4.11. *f is a fixed point.*

Proof. Lemmas 4.9 and 4.10 \square

This suffices to prove the completeness component so we have our desired soundness and completeness theorem by combining Theorems 4.6 and 4.12.

Theorem 4.12 (Completeness of ProbKF^{ω}). *If $\mathcal{A} \models \text{ProbKF}^{\omega}$ then there is an probabilistic modal structure \mathcal{M} , fixed point f and $w \in W$ such that $\mathcal{A} = \text{IndMod}_{\mathcal{M}}[f, w]$.*

Proof. Take $\mathcal{A} \models \text{ProbKF}^{\omega}$. Take \mathcal{M} and f as in Definition 4.7 and use Corollary 4.11 and Lemma 4.8 to check that \mathcal{M} is in fact a probabilistic modal structure and that f is a fixed point. Take $w = \text{Theory}(\mathcal{A})$, which is a ProbKF^{ω} -consistent set of formulas by Theorem 4.6. Then by Lemma 4.9 we have that for all φ , $\text{IndMod}_{\mathcal{M}}[f, w] \models \varphi \iff \mathcal{A} \models \varphi$. Since $\text{IndMod}_{\mathcal{M}}[f, w]$ and \mathcal{A} also have the same domain, namely \mathbb{N} , this suffices to show that $\mathcal{A} = \text{IndMod}_{\mathcal{M}}[f, w]$. \square

One might hope to be able to produce a theorem of the form:

Desired Theorem 4.12.1. *Let \mathcal{A} be a model of $L_{P, \Gamma}$ which interprets the arithmetic vocabulary by the standard model. Then: $\mathcal{A} \models \text{ProbKF}$ if and only if there is an probabilistic structure \mathcal{M} , fixed point f and $w \in W$ such that $\mathcal{A} = \text{IndMod}_{\mathcal{M}}[f, w]$.*

This theorem would show that ProbKF is an \mathbb{N} -categorical axiomatisation in the sense of (Fischer, Halbach, Kriener, & Stern, n.d.). However proving such a theorem remains a challenge. If one takes W to contain all the maximally ProbKF -consistent sets then we can only prove $\text{IndMod}_{\mathcal{M}}[f, w] \models \text{KF} \cup \text{InteractionAx}$ for w ω -consistent²¹ and so we wouldn’t be able to use

²¹We only need ω -consistency instead of arithmetical soundness since $\text{KF} \cup \text{InteractionAx}$ is Π_2^0 . The other use of Lemma 4.9 was to show $\mathcal{A} = \text{IndMod}_{\mathcal{M}}[f, \text{Theory}(\mathcal{A})]$ but if we just consider standard \mathcal{A} then the required assumptions on $w = \text{Theory}(\mathcal{A})$ have already been made.

Lemma 4.10 to derive the conclusion that f is in fact a fixed point. If instead one takes W to be just the ω -consistent, maximally ProbKF-consistent sets of sentences then one would not be able to prove Lemma 4.8 since it would not be the case that $[n] \subseteq [k] \implies \mu[n] \leq \mu[k]$ since we would not be able to conclude from the fact that $[n] \subseteq [k]$ the fact that $\top^{\ulcorner n \urcorner} \rightarrow \top^{\ulcorner k \urcorner}$ so we cannot use Item 23.

5 Conclusions

In this paper we have presented a construction of a semantics for a language which includes sentences that can talk about their own probabilities and have given a corresponding axiomatic theory. The semantics are given by applying a familiar construction of a semantics for type-free truth, namely Kripke’s construction from (Kripke, 1975), to possible world style structures. In this semantics some sentences are only assigned ranges of probability values instead of a single value but this will only happen for “problematic” sentences, in most cases sentences we consider will be grounded then they are assigned a particular probability value and one can reason in a fairly natural way. We also observed that this semantics can also allow for σ -additivity. We provided an axiomatisation which allows one to reason about these semantics in a clear way and show, for example, which assumptions about probability would lead to inconsistencies.

We showed that if one expresses introspection principles by using a truth predicate to do the job of quotation and disquotation these principles are consistent with the framework. Although we have only considered introspection principles here we believe the phenomena is quite general. For evidence of this we can see in (Stern, 2014a) and (Stern, 2014b) that the strategy worked well in the case of necessity. In future work we would like to investigate exactly how one should express principles and show that this allows one to avoid the paradoxical contradictions.

One limitation of this construction is that it does not yet have the ability to account for conditional probabilities. Furthermore it is not clear how one could add conditional probabilities and give a definition of $(w, f) \models_{\mathcal{M}}^{SKP} P_{\geq \alpha}(\ulcorner \varphi \urcorner \mid \ulcorner \psi \urcorner)$ in the style of strong Kleene three valued logic which can deal with them.

An alternative semantics for probability can be constructed if we consider applying different semantics for truth to probabilistic modal structures. In current work we have so far considered two different alternative semantics, one based on supervaluational logic and the other a revision semantics. If we apply a similar construction to the one presented here except base it on a supervaluational scheme instead of strong Kleene’s three valued logic we result in a notion of probability which can be seen as imprecise probability. We believe that this leads to a philosophically interesting connection. Additionally if we use a supervaluational scheme our language need not be as fixed as it is here, in particular the semantics can also apply to conditional probabilities. We can also consider revision theories of probability which provide a semantics which is perhaps the most conservative to our usual ideas of probability because it assigns single

probability values to each sentence. The downside of this semantics is that the construction will never terminate so there is no final semantics to consider, just a sequence of continually improving semantics.

A particularly interesting principle involving conditional probabilities is Miller's principle, or a principle of trust, which says

$$P_{\geq \alpha}^A (\ulcorner \varphi \urcorner \mid \ulcorner P_{\geq \alpha}^B \urcorner \ulcorner \varphi \urcorner \urcorner)$$

This is for example the form of Lewis's principal principle, van Fraassen's reflection principle, self-trust, or just A taking B to be an expert. In this form it is inconsistent with probabilism. In both these constructions we can show that when expressed using a truth predicate trust is satisfied in exactly the structures where the operator version is satisfied. This is when trust is expressed by:

$$P_{\geq \alpha}^A (\ulcorner T \urcorner \ulcorner \varphi \urcorner \urcorner \mid \ulcorner P_{\geq \alpha}^B \urcorner \ulcorner \varphi \urcorner \urcorner)$$

This is more evidence that the idea of expressing principles using the truth predicate is a good one.

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