Rational Probabilistic Incoherence? A Reply to Michael Caie

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In Michael Caie's paper 'Rational Probabilistic Incoherence'[Caie, 2013], Caie argues that it is sometimes rational to have probabilistically incoherent credences and therefore that probabilism is false. Our paper provides further analysis of his arguments.

Caie first observes that if probabilism holds then an agent is rationally required to have poor epistemic access to some of her own credences. He presents this as a prima facie problem for probabilism. In Section 1 we shall show that assumptions weaker than probabilism lead to the requirement of poor epistemic access and, given one way of understanding epistemic access, only very basic requirements on credences are needed for the result. This all suggests that probabilism is not to blame for the failure of rational introspection.

Caie admits that this is merely a *prima facie* problem for probabilism. He says:

> We might simply accept this consequence [the rational requirement that an agent have poor epistemic access to her own credences] of probabilism despite its prima facie implausibility. In this essay, I'll argue that this isn't the right response. *To this end* I'll show that [a situation used in the prima facie problem for probabilism] can be used to expose flaws in the accuracy-dominance argument for probabilism. Once these flaws are exposed, we can see that considerations of accuracy, instead of motivating probabilism support the claim that a rational agent's credences should be probabilistically incoherent. ([Caie, 2013, pg. 528]; our emphasis)

One might therefore suppose that Caie's modified accuracy dominance criterion would not require an agent to sometimes have poor epistemic access to her own credences. But we show in Section 2 that this is not the case. This observation doesn't weaken his argument for rational probabilistic incoherence but instead throws light on the connection between Caie's two arguments.

In Sections 3.1 and 3.2 we point out some previously unnoticed features of Caie's proposed accuracy criterion. In Section 3.1 we show that the credences

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which are rationally required according to Caie's proposed accuracy criterion lead to sure loss on a particular bet and that furthermore this sure loss could be minimized by holding other credences. This is an undesirable feature and shows that Caie cannot use Dutch book considerations to support his criterion. Caie shows in his paper that Dutch book considerations in fact don't support probabilism so we cannot avoid the conflict between accuracy and Dutch book considerations by rejecting Caie's modification of the accuracy criterion, instead we should see this as an undesirable feature that has to be dealt with.

In Section 3.2 we show that the criterion is dependent on the scoring rule on which it is based. Although different choices of scoring rules all still lead to rational probabilistic incoherence, so this again doesn't weaken Caie's argument for rational probabilistic incoherence, they lead to different credal states being deemed rational. Therefore one needs to do more work to explain what rational constraints accuracy-domination considerations lead to if Caie's modification is accepted. At the end of Section 3.2 we present some options for answers to this question.

1 A problem for Caie's prima facie argument against probabilism

The principle which Caie argues against is:

Probabilism: *An agent is rationally required to have probabilistically coherent credences.*

where an agent's credences, Cr, are probabilistically coherent if they satisfy:

- For any necessary proposition,[1] $\top$, $Cr(\top) = 1$
- For any proposition, φ, $0 \leqslant Cr(\varphi)$
- If φ and ψ are incompatible propositions, i.e. it is necessary that $\neg(\varphi \wedge \psi)$, then $Cr(\varphi \vee \psi) = Cr(\varphi) + Cr(\psi)$

The two arguments which Caie gives against Probabilism both rely on a sentence whose truth depends on an agent's credences. This sentence is #, which is introduced by stipulating that the name '#' refers to the following, interpreted, sentence:

> Hiro's credence in the proposition expressed by # isn't greater than or equal to 0.5.

He uses 'Cr_H' to abbreviate 'Hiro's credence in' and 'ρ' to abbreviate 'the proposition expressed by'. The above can then be represented by:

(#) $$\neg\, Cr_H \rho(\#) \geqslant 0.5$$

[1] Caie originally stated this condition as "For any logical truth". However, he cannot take the necessary propositions to be logical truths because he requires the proposition expressed by # to be the same as that expressed by '$\neg\, Cr\rho(\#) \geqslant 0.5$', even though these are not logically equivalent.

In this paper we take for granted the existence of such propositions.

The first argument which Caie gives against probabilism relies on the following theorem.

Theorem 1. *If Hiro's credences are probabilistically coherent then, given the existence of #, one of the following must hold:*

Neg Insensitivity: $\neg\, Cr_H \rho(\#) \geqslant 0.5$ *and* $\neg\, Cr_H \rho(\text{`}\neg\, Cr_H \rho(\#) \geqslant 0.5\text{'}) > 0.5$

Pos Insensitivity: $Cr_H \rho(\#) \geqslant 0.5$ *and* $\neg\, Cr_H \rho(\text{`}Cr_H \rho(\#) \geqslant 0.5\text{'}) > 0.5$

Proof. One consequence of probabilism is:

Propositions: *A single proposition cannot be assigned different credences.*

By assumption $\rho(\#) = \rho(\text{`}\neg\, Cr_H \rho(\#) \geqslant 0.5\text{'})$, so using Propositions, $Cr_H \rho(\#) = Cr_H \rho(\text{`}\neg\, Cr_H \rho(\#) \geqslant 0.5\text{'})$. Using this we can see that either $\neg\, Cr_H \rho(\#) \geqslant 0.5$ and $\neg\, Cr_H \rho(\text{`}\neg\, Cr_H \rho(\#) \geqslant 0.5\text{'}) \geqslant 0.5$ holds, or $Cr_H \rho(\#) \geqslant 0.5$ and $Cr_H \rho(\text{`}\neg\, Cr_H \rho(\#) \geqslant 0.5\text{'}) \geqslant 0.5$ holds. Neg Insensitivity immediately follows from the former. Pos Insensitivity follows from the latter using another consequence of Probabilism, which we call Negation.

Negation: $Cr_H \rho(\text{`}\neg A\text{'}) \geqslant 0.5 \implies \neg\, Cr_H \rho(\text{`}A\text{'}) > 0.5.$

□

Caie argues "it is prima facie implausible that an agent could be rationally required to have poor epistemic access to her own credal state"[Caie, 2013, pg. 4]. He therefore supports the following principle:

Weak Rational Introspection: *An agent should not be rationally required to have poor epistemic access to her own credal state.*

If either Neg Insensitivity or Pos Insensitivity hold then Hiro is insensitive to his own credences. So to maintain Weak Rational Introspection, Probabilism must be rejected.

The only components of Probabilism required for this proof were Propositions and Negation, which are much more innocent than Probabilism.[2] It would be particularly difficult to reject Propositions since it is a basic assumption about credences. In fact Caie supports Propositions throughout his paper. Instead he rejects Negation and claims there are situations where a rational agent should have credence $Cr_H \rho(\text{`}\neg A\text{'}) \geqslant 0.5$ whilst still holding that $Cr_H \rho(\text{`}A\text{'}) > 0.5$. This is itself prima facie implausible, perhaps more so than the requirement to have poor epistemic access to some of one's own credal states.

Even rejecting Negation does not clearly allow one to avoid the requirement to have poor epistemic access to ones own credal state. This is because in the proof of Theorem 1 we only used Negation to derive Pos Insensitivity from

[2] For example a Dempster-Shafer belief function will satisfy Negation and Propositions.

Alternative Pos Insensitivity: $Cr_H\rho(\#)\geqslant 0.5$ *and* $Cr_H\rho(`\neg Cr_H\rho(\#)\geqslant 0.5') \geqslant 0.5$

So by just using Propositions we could already conclude that either Neg Insensitivity or Alternative Pos Insensitivity hold. If Hiro satisfies Alternative Pos Insensitivity then although he is in some credal state, $Cr_H\rho(\#)\geqslant 0.5$, he has high credence that he is not in this credal state. One might therefore take Alternative Pos Insensitivity to be an instance of Hiro being insensitive to his own credences. Under that assumption Weak Rational Introspection and Propositions conflict with one another and therefore Weak Rational Introspection would have to be rejected since Propositions is much more plausible than Weak Rational Introspection.

One might instead reject that Alternative Pos Insensitivity is an instance of Hiro being insensitive to his own credences, perhaps by arguing that our intuition that Alternative Pos Insensitivity displays insensitivity relies on an implicit acceptance of Negation. However even if Caie was to do this, Weak Rational Introspection will still be problematic for him. This is because the way which Caie argues that accuracy considerations should apply will lead to the failure of Weak Rational Introspection.

2 Accuracy Domination Arguments Lead to the Failure of Weak Rational Introspection

The argument against probabilism which Caie most strongly supports is that accuracy-dominance considerations in fact lead to rational probabilistic incoherence. We now show that his accuracy-dominance criterion conflicts with Weak Rational Introspection.

The accuracy argument assumes the goal of an agent is to have credences which are as accurate as possible. Caie argues that we should only consider how accurate a credal state would be *if they were the agent's credences*. So for propositions whose truth value is determined by an agent's credences, such as $\rho(\#)$, the only relevant measure of accuracy is the accuracy of the credal state at the world where the agent has that credal state. Since all our examples are based on cases where the credal state is only defined over such propositions we just state the criterion for these cases.

Caie focuses on the Brier score, BS, as a measure of the accuracy of an agent's credal states at a world and we shall initially do the same.

Caie's Accuracy Criterion: *Let $\varphi_1 \dots \varphi_n$ be a finite algebra*[3] *of distinct propositions whose truth depends only on what the agent's credences are. Let Cr be some credal state i.e. an assignment of a real number to each proposition.*[4]

[3] Namely a collection of propositions which is closed under negations and finite disjunctions. We ignore $\top$ and $\bot$ since we can assume that these will always be assigned credences 1 and 0 respectively.

[4] We use 'Cr_H' as an abbreviation for 'Hiro's credence in', so the function from propositions to reals which it denotes depends on the situation. a, b, c and d are used to rigidly denote functions from propositions to reals. We use Cr as the variable which might be instantiated by such a, b etc.

Define

$$\mathsf{U}(Cr) := 1 - \mathsf{BS}(w_{Cr}, Cr) = 1 - \frac{1}{n}\sum_{i=1}^{n}(w_{Cr}(\varphi_i) - Cr(\varphi_i))^2$$

where w_{Cr} is the world that would be actual if Cr were the agent's credences.

If credal state Cr is such that for all other credal states Cr', $\mathsf{U}(Cr) > \mathsf{U}(Cr')$, then an agent is rationally required to hold credal state Cr.

In the case of Hiro this criterion leads to $Cr\rho(\#) = 0.5$ and $Cr\rho(\neg\#) = 1$ as the rationally required credal state. Since this is a probabilistically incoherent credal state Caie concludes that accuracy considerations lead to the rejection of Probabilism.

We can now present our example which shows that Caie's proposal conflicts with Weak Rational Introspection.

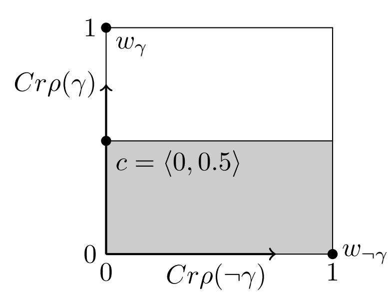
Instead of considering # we consider γ which is a minor modification of #. Let 'γ' name the following sentence:

June's credence in the proposition expressed by γ isn't greater than 0.5.

This can be represented by:

(γ) $$\neg\, Cr_J\rho(\gamma) > 0.5$$

We can represent this situation diagrammatically as Caie does in his paper. Since the only propositions we are interested in are $\rho(\gamma)$ and $\rho(\neg\gamma)$ we shall just consider June's credences in these two propositions. We can therefore represent June's possible credences by points on the graph whose axes are $Cr\rho(\gamma)$ and $Cr\rho(\neg\gamma)$. The two possible states of affairs are w_γ, where γ is true, and $w_{\neg\gamma}$, where γ is false. As Caie does we shall also identify these as points in the diagram, as labelled below, and we shall only focus on the credal states in $[0,1]^2$. Unlike Caie we also shade the credal states where if they were June's credences then γ would be true.



Theorem 2. *According to Caie's Accuracy Criterion June is rationally required to be in state c.*

Proof.

$$\mathsf{U}(Cr) = \begin{cases} 1 - \mathsf{BS}(w_{\neg\gamma}, Cr) & \text{if } Cr\rho(\gamma) > 0.5 \\ 1 - \mathsf{BS}(w_{\gamma}, Cr) & \text{otherwise} \end{cases}$$

$$Cr\rho(\gamma) > 0.5 \implies \mathsf{BS}(w_{\neg\gamma}, Cr) = \frac{Cr\rho(\gamma)^2 + (1 - Cr\rho(\neg\gamma))^2}{2} > \frac{0.5^2}{2} = 0.125$$

$$\neg Cr\rho(\gamma) > 0.5 \implies \mathsf{BS}(w_{\gamma}, Cr) = \frac{(1 - Cr\rho(\gamma))^2 + Cr\rho(\neg\gamma)^2}{2} \geqslant \frac{(1-0.5)^2}{2} = 0.125$$

So the minimal $\mathsf{U}(Cr)$ possible is 0.125. One can check that this is obtained only at c. □

Credal state c has the properties $\neg\, c\rho(\gamma) > 0.5$ and $\neg\, c\rho(\text{`}\neg\, Cr_J\rho(\gamma) > 0.5\text{'}) > 0.5$. This is directly analogous to Neg Insensitivity as we stated it above.[5] Therefore c is a credal state where June has poor epistemic access to her own credences. According to Caie's accuracy criterion c is a rationally required credal state. This shows that Caie's Accuracy Criterion is inconsistent with Weak Rational Introspection even if Alternative Pos Insensitivity is not taken to be an instance of having poor epistemic access to one's own credences. One can deal with this inconsistency either by rejecting Caie's Accuracy Criterion or Weak Rational Introspection, though it is important to note that this does not weaken an argument against Probabilism since each of Caie's Accuracy Criterion and Weak Rational Introspection lead to rejecting Probabilism. The inconsistency instead shows that accuracy considerations do not support the prima facie plausible principle of Weak Rational Introspection.

In the next section we will present some features of Caie's Accuracy Criterion which were previously unnoticed.

3 Features of Caie's Accuracy Criterion

3.1 Caie's Accuracy Criterion leads to needless loss

As Caie mentions in his paper sometimes an agent will always value as fair a set of bets which guarantee a loss of money. He says that in such situations Dutch Book considerations should require an agent to minimize her loss. However, we show here that there are some situations where Caie's Accuracy Criterion leads to an agent being rationally required to hold credal states which lead to needless loss. This shows that Caie cannot use Dutch Book considerations to support his criterion.

Caie says that Dutch Book considerations should lead one to accept Loss Minimization:

Loss Minimization: *An agent is rationally required to have a credal state which, assuming she bets with her credences, minimizes her possible losses.*

[5] We might write Neg Insensitivity more generally as: Cr_J has Neg Insensitivity if there is some proposition φ and interval Δ such that $\neg\, Cr_J\varphi \in \Delta$ and $\neg\, Cr_J\rho(\text{`}\neg\, Cr_J\varphi \in \Delta\text{'}) > 0.5$. c has Neg Insensitivity in this sense.

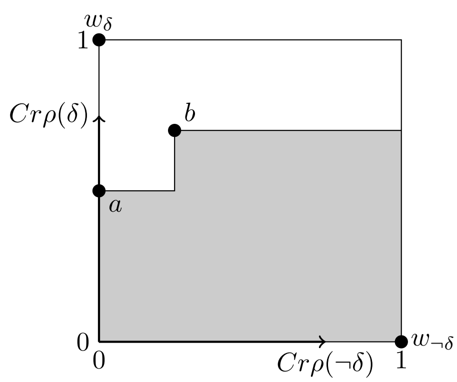
We shall show that Caie's Accuracy Criterion is incompatible with a weaker principle, namely Weak Loss Minimization.

Weak Loss Minimization: *An agent should not be rationally required to have a credal state which, assuming she bets with her credences, does not minimize her possible losses.*

Consider the following, admittedly unusual,[6] sentence about Roy's credences:

$$(\delta) \qquad Cr_R\rho(\delta) \leqslant 0.5 \vee (Cr_R\rho(\delta) \leqslant 0.7 \wedge Cr_R\rho(\neg\delta) \geqslant 0.25)$$

As before, we can represent this situation diagrammatically.



$a = \langle 0, 0.5\rangle$
$b = \langle 0.25, 0.7\rangle$

Theorem 3. *According to Caie's Accuracy Criterion Roy is rationally required to be in credal state b, which is a credal state which, assuming Roy bets with his credences, will not minimize his possible losses. Therefore Caie's Accuracy Criterion and Weak Loss Minimization are incompatible principles.*

Proof. Consider credal states a and b, and the following set of bets:

{[-\$1 if δ, \$0 otherwise], [\$1 if $\neg\delta$, \$0 otherwise]}

Assuming Roy bets with his credences he will be willing to pay $-\$Cr_R\rho(\delta) + \$Cr_R\rho(\neg\delta)$ for this set of bets. Whether Roy is in credal state a or b, δ will be true so he will loose the wager. The total loss on this wager is therefore $1 - Cr_R\rho(\delta) + Cr_R\rho(\neg\delta)$. For a and b this is:

a: $1 - 0.5 + 0 = \$0.5$

b: $1 - 0.7 + 0.25 = \$0.55$

[6]We could instead consider a proposition A where Roy's credence in $A \leftrightarrow Cr_R A \leqslant 0.5 \vee (Cr_R A \leqslant 0.7 \wedge Cr_R A \geqslant 0.25)$ is high instead of requiring that this holds of necessity. Then we could modify our results to see that the rationally required credences have expected losses which could be minimized, and the expectedly most accurate credal state differs depending on whether the Brier score or logarithmic score are used. For simplicity we shall not do this.

So we see that if Roy holds credal state b he fails to minimize his possible loss. Therefore by Weak Loss Minimization Roy should not be rationally required to be in credal state b.

We now show that b is rationally required according to Caie's Accuracy Criterion. We only give the argument explicitly for a and b, since these are the states which are competing for being the most accurate and minimizing the loss on this bet.

$$\mathsf{U}(Cr) = \begin{cases} 1 - \mathsf{BS}(w_\delta, Cr) & \text{if } Cr\rho(\delta) \leqslant 0.5 \vee (Cr\rho(\delta) \leqslant 0.7 \wedge Cr\rho(\neg\delta) \geqslant 0.25) \\ 1 - \mathsf{BS}(w_{\neg\delta}, Cr) & \text{otherwise} \end{cases}$$

$$\mathsf{BS}(w_\delta, a) = \frac{0^2 + 0.5^2}{2} = 0.125$$

$$\mathsf{BS}(w_\delta, b) = \frac{0.25^2 + 0.3^2}{2} = 0.07625$$

Therefore $\mathsf{U}(a) < \mathsf{U}(b)$. By checking that all other credal states Cr have $\mathsf{U}(Cr) < \mathsf{U}(b)$,[7] we can see that credal state b is rationally required according to Caie's Accuracy Criterion. □

This shows that Caie's Accuracy Criterion is not compatible with Weak Loss Minimization. Although this is an undesirable feature of Caie's Accuracy Criterion it does not give us a reason to reject the modification of the criterion since the original accuracy criterion also conflicts with Weak Loss Minimization.[8] It is nonetheless interesting that accuracy considerations and Dutch Book considerations conflict and it is a consequence of Caie's Accuracy Criterion which one should be aware of.

3.2 Scoring Rule Dependence of Caie's Accuracy Criterion

Not only does Caie's Accuracy Criterion conflict with Dutch Book considerations, it also conflicts with criteria analogous to Caie's Accuracy Criterion which are based on different scoring rules. This is in contrast to the usual formulation of the accuracy criterion.

We can use the above example to show that the criterion differs depending on whether the Brier score or the logarithmic score are used.

Theorem 4. *Roy is rationally required to be in credal state a, according to the criterion analogous to Caie's Accuracy Criterion but based on the logarithmic scoring rule.*[9]

[7]Which can be seen because BS is truth-directed.

[8]Which can be seen because Weak Loss Minimization leads to rational probabilistic incoherence whereas the usual accuracy criterion does not.

[9]One might think that the logarithmic scoring could be compatible with Weak Loss Minimization. But this is not the case.

Proof.

$$\mathsf{U}(Cr) = \begin{cases} 1 - L(w_{\neg\delta}, Cr) & \text{if } Cr\rho(\delta) \leqslant 0.5 \vee (Cr\rho(\delta) \leqslant 0.7 \wedge Cr\rho(\neg\delta) \geqslant 0.25) \\ 1 - L(w_{\delta}, Cr) & \text{otherwise} \end{cases}$$

We only need to explicitly consider a and b since it is clear that any other $\mathsf{U}(Cr)$ will be lower than either $\mathsf{U}(a)$ or $\mathsf{U}(b)$.

$$\mathsf{U}(a) = 1 - L(w_{\delta}, a) = 1 + \log(0.7) + \log(1 - 0.25) \approx 0.36$$
$$\mathsf{U}(b) = 1 - L(w_{\delta}, b) = 1 + \log(0.5) + \log(1 - 0) \approx 0.30$$

Therefore basing Caie's Accuracy Criterion on the logarithmic scoring rule leads to Roy being rationally required to be in credal state a. □

In Theorem 4 we showed that Caie's Accuracy Criterion based on the Brier score led to Roy being rationally required to have credal state b. Therefore we see that Caie's Accuracy Criterion is dependent on the scoring rule chosen.

This is not specific to the logarithmic scoring rule. We can find similar examples for most pairs of scoring rules.[10]

Given this scoring rule dependence, if one accepts Caie's modification to the accuracy criterion one needs to explain what rational constraints accuracy-dominance considerations lead to. There are at least four options.[11] Firstly, one could give arguments for one particular scoring rule and argue that accuracy-dominance considerations require one to minimize inaccuracy with respect to that scoring rule. Secondly, one could take a *subjectivist* approach and argue that for each agent and context there is some particular measure of inaccuracy which is appropriate. Thirdly, one could take a *supervaluationist* approach and argue that the notion of inaccuracy is vague and that any inaccuracy measure satisfying certain conditions is an appropriate precisification of it; to satisfy accuracy dominance considerations one would then have to minimise inaccuracy with respect to at least one appropriate inaccuracy measure. Lastly one could take an *epistemicist* approach and argue that although there is some particular scoring rule which one should be minimising inaccuracy with respect to, we do not know which it is.[12] Each of these ways of dealing with the scoring rule dependence will still lead to the rejection of probabilism since different scoring rules will still lead to rational probabilistic incoherence.

[10] This can be seen by varying the above proof. Choose credal states a and b, where a is closer to $\langle 0, 1\rangle$ under one scoring rule, and b is closer under the other. Then construct a sentence where a and b are the credal states which compete for being the most accurate.

[11] This problem is very closely related to a problem for the traditional accuracy argument which is that there is no credence function that dominates on every measure. This is discussed in [Pettigrew, 2011, section 6.2.2]. Furthermore the ways of dealing with the two problems are similar and these options presented here parallel the options presented in Pettigrew's article.

[12] The disadvantage of this version of accuracy considerations is that an agent does not have the resources to know whether she satisfies the rational requirement or not.

4 Conclusion

In Section 1 we showed that Probabilism isn't to blame for the failure of Weak Rational Introspection, therefore undermining Caie's first argument for rational probabilistic incoherence. In Section 2 we showed that Weak Rational Introspection also conflicts with accuracy considerations. So accuracy considerations do not show that the correct response to the conflict between Weak Rational Introspection and Probabilism is to retain Weak Rational Introspection. However this does not mean that one should still keep Probabilism since accuracy considerations also lead to rejecting Probabilism.

In Section 3.1 and Section 3.2 we showed some surprising features of Caie's Accuracy Criterion, namely that it leads to needless loss and is dependent on the scoring rule chosen. However these observations do not weaken Caie's argument for rational probabilistic incoherence because Dutch Book considerations and Caie's Accuracy Criterion based on different scoring rules also lead to rational probabilistic incoherence. Moreover our argument in Section 3.1 that accuracy considerations conflict with Dutch Book considerations cannot be taken as an argument against Caie's modified accuracy criterion since the traditional accuracy criterion has the same feature.

References

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