# Q & A Session: Bonus Material

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- Sharon Ryan (now on website) gives an argument *for* (CB) as a rational requirement, which uses these three premises.

  **The Closure of Rational Belief Principle** (CRBP).
  If $S$ rationally believes $p$ at $t$ and $S$ knows (at $t$) that $p$ entails $q$, then it would be rational for $S$ to believe $q$ at $t$.

  **The No Known Contradictions Principle** (NKCP).
  If $S$ knows (at $t$) that $\bot$ is a logical contradiction, then it would *not* be rational for $S$ to believe $\bot$ (at $t$).

  **The Conjunction Principle** (CP).
  If $S$ rationally believes $p$ at $t$ and $S$ rationally believes $q$ at $t$, then it would be rational for $S$ to believe $\ulcorner p \,\&\, q \urcorner$ at $t$.

- Ryan's (CRBP) & (NKCP) have analogues in our framework (which *are* coherence requirements). But, (CP) does *not*.

(SPC) If $p \vDash q$, then any $\mathbf{B}$ s.t. $\{B(p), D(q)\} \subseteq \mathbf{B}$ is incoherent.

(NCB) Any $\mathbf{B}$ such that $\{B(\bot)\} \subseteq \mathbf{B}$ is incoherent.

¬(CP) *Not* every $\mathbf{B}$ s.t. $\{B(p), B(q), D(p \,\&\, q)\} \subseteq \mathbf{B}$ is incoherent.





- Some (*e.g.*, Duddy and Piggins) worry that our naïve (Hamming) measure of inaccuracy "double-counts".

  > The most widely used metric in the literature is the Hamming metric. This is simply the number of propositions over which the two individuals disagree. So the distance between $\{B(p), B(q), B(p \,\&\, q)\}$ and $\{B(p), D(q), D(p \,\&\, q)\}$ is 2. But therein lies the problem. The proposition $\neg(p \,\&\, q)$ is a logical consequence of $p$ and $\neg q$, and $p \,\&\, q$ is a logical consequence of $p$ and $q$. So, given that the individuals both accept $p$, the disagreement over $p \,\&\, q$ is implied by the disagreement over $q$. The Hamming metric appears to be double counting because it ignores the fact that the propositions are logically interconnected.

- One might have thought that the robustness of our result $(\mathcal{R}) \Rightarrow$ (WADA) allows us to sidestep this problem.
- However, no constant/rigid weighting scheme + additive distance measure can (generally) accommodate these types of "relative informativeness" relations among propositions.





- People who voice the "double counting" worry tend to presuppose that *deductive cogency* is a rational requirement. In particular, they tend to presuppose:

(MPC) If $\{p_1, \ldots, p_n\}$ entails $q$, then any belief set $\mathbf{B}$ containing $\{B(p_1), \ldots, B(p_n), D(q)\}$ is *epistemically incoherent*.

- We call this (MPC), because it is similar to *multi-premise closure*. Of course, we *reject* (MPC). However, we *accept*:

(SPC) If $p$ entails $q$, then any belief set $\mathbf{B}$ containing $\{B(p), D(q)\}$ is *epistemically incoherent*.

- (SPC) follows from (WADA). So, *some* degree of sensitivity to "relative informativeness" emerges from our approach.
- We think this is *the right amount* of sensitivity to "relative informativeness." So, we are not too bothered by the DCW.
- It is an open question whether there is a way of defining distance such that (MPC) follows from (WADA) [or $(\mathcal{R})$].

- Briggs, Cariani, Easwaran & Fitelson (now on website) apply "coherence" to *aggregation paradoxes*. Coherence *can* fail to be preserved by majority rule, but only on weird agendas.
- Consider a language w/16 state descriptions $s_1, \ldots, s_{16}$. Let:

$$p \stackrel{\text{def}}{=} s_1 \vee s_2 \vee s_3 \vee s_4 \qquad q \stackrel{\text{def}}{=} s_1 \vee s_5 \vee s_6 \vee s_7$$
$$r \stackrel{\text{def}}{=} s_2 \vee s_5 \vee s_8 \vee s_9 \qquad s \stackrel{\text{def}}{=} s_3 \vee s_6 \vee s_8 \vee s_{10}$$
$$t \stackrel{\text{def}}{=} s_4 \vee s_7 \vee s_9 \vee s_{10} \qquad \Sigma \stackrel{\text{def}}{=} \{p, q, r, s, t\}$$

(i) Any two sentences in $\Sigma$ are logically consistent.
  - because any pair shares a state description.

(ii) Any three sentences in $\Sigma$ are logically inconsistent.
  - because every state description occurs exactly twice.

(iii) Any four sentences in $\Sigma$ are coherent (if jointly believed).
  - Non-dominance is ensured by the fact that some such judgment sets will *fail* to contain a subset $\beta$ such that, at every world, a majority of $\beta$'s members are inaccurate.

(iv) $\Sigma$ is incoherent (if jointly believed: $B(\Sigma)$).
  - At every $w$, most of $B(\Sigma)$'s members are inaccurate.





| | $p$ | $q$ | $r$ | $s$ | $t$ |
|---|---|---|---|---|---|
| $J_1$ | $B$ | $B$ | $B$ | $B$ | $D$ |
| $J_2$ | $B$ | $B$ | $B$ | $D$ | $B$ |
| $J_3$ | $B$ | $B$ | $D$ | $B$ | $B$ |
| $J_4$ | $B$ | $D$ | $B$ | $B$ | $B$ |
| $J_5$ | $D$ | $B$ | $B$ | $B$ | $B$ |
| Majority | $B$ | $B$ | $B$ | $B$ | $B$ |

- Each judge can be *coherent* because judgment sets with 4/5 beliefs (and 1/5 disbeliefs) over $\Sigma$ can be *non*-dominated.
- This is because there will be worlds in which a majority of such judgments are accurate. (For example: in worlds that make state description $s_1$ true, $p$, $q$ and $\neg t$ are all true.)
- However the (80%!) majority believes *all* members of $\Sigma$. And, any judgment set containing these judgments *must be dominated*. So, *majority rule doesn't preserve coherence*. □
- On the next slide, we'll sketch a proof of our positive JA-Theorem. The key will be to use $(\mathcal{R}) \Rightarrow$ (WADA).





- Our main result for full belief implies that if **B** is representable by some Pr-function *via* a "strict $\frac{1}{2}$-threshold," then **B** must be coherent (*viz.*, non-dominated).
- For majority acceptance on individually consistent and complete inputs this is clearly true. The probability function in question is just the pattern of individual votes:

$$\text{For all } p, \Pr^{\star}(p) \stackrel{\text{def}}{=} \frac{\text{\# of judges for } p}{\text{\# of total judges}}.$$

- To verify this, note that $\Pr^{\star}(\cdot)$ satisfies the Pr-axioms. Additivity is the only axiom that deserves comment.
  - Suppose $p$, $q$ are m.e. If $p$ is accepted by $\frac{r}{y}$ of the judges and $q$ is accepted by $\frac{s}{y}$ of the judges, then (by consistency + completeness) $p \vee q$ will be accepted by $\frac{r+s}{y}$ of the judges.
- $\therefore$ By our main result and the existence of $\Pr^{\star}(\cdot)$, it follows that majority rule on consistent and complete profiles always yields *coherent* aggregations. That is, if judges satisfy (CB), then their majority aggregate satisfies (WADA).





- Recall, our axioms for *comparative probability* (which we called $(\mathfrak{C}_2)$) were as follows (where $p \succeq q \stackrel{\text{def}}{=} p \succ q \vee p \sim q$).

**Totality**. $(p \succeq q) \vee (q \succeq p)$.

**Transitivity**. If $p \succeq q$ and $q \succeq r$, then $p \succeq r$.

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(A1) $\top \succ \bot$.

(A2) If $p \vDash q$, then $q \succeq p$.

(A5) If $\langle p, q \rangle$ and $\langle p, r \rangle$ are mutually exclusive, then:

$$q \succeq r \iff (p \vee q) \succeq (p \vee r).$$

- de Finetti conjectured that these axioms were sufficient to ensure *full* representability of $\succeq$ by a probability function $(\mathfrak{C}_4)$.
- de Finetti reported that there are no $(\mathfrak{C}_4)$-counterexamples involving algebras $\mathcal{B}_n$ containing $n \leq 4$ states. [This is non-trivial to do by hand, but easy with today's computers.]
- Interestingly, there are $(\mathfrak{C}_3)$-counterexamples when $n \geq 5$. This was discovered several years later by Kraft *et. al.*.

- We won't write down the entire Kraft *et. al.* ordering $\succeq$ as it involves a complete ranking of 32 propositions. Instead, we focus only the following, salient 8-proposition fragment.

| $\succeq$ | $\mathfrak{s}_1$ | $\mathfrak{s}_2 \vee \mathfrak{s}_4$ | $\mathfrak{s}_3 \vee \mathfrak{s}_4$ | $\mathfrak{s}_1 \vee \mathfrak{s}_2$ | $\mathfrak{s}_2 \vee \mathfrak{s}_5$ | $\mathfrak{s}_1 \vee \mathfrak{s}_4$ | $\mathfrak{s}_1 \vee \mathfrak{s}_2 \vee \mathfrak{s}_4$ | $\mathfrak{s}_3 \vee \mathfrak{s}_5$ |
|---|---|---|---|---|---|---|---|---|
| $\mathfrak{s}_1$ | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\mathfrak{s}_2 \vee \mathfrak{s}_4$ | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\mathfrak{s}_3 \vee \mathfrak{s}_4$ | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 |
| $\mathfrak{s}_1 \vee \mathfrak{s}_2$ | 1 | 1 | 0 | 1 | 0 | 0 | 0 | 0 |
| $\mathfrak{s}_2 \vee \mathfrak{s}_5$ | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 |
| $\mathfrak{s}_1 \vee \mathfrak{s}_4$ | 1 | 1 | 1 | 1 | 0 | 1 | 0 | 0 |
| $\mathfrak{s}_1 \vee \mathfrak{s}_2 \vee \mathfrak{s}_4$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\mathfrak{s}_3 \vee \mathfrak{s}_5$ | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 1 |

- This example satisfies ($\mathfrak{C}_2$), but violates ($\mathfrak{C}_3$).
- Dana Scott gave necessary and sufficient conditions for full Pr-representability ($\mathfrak{C}_4$); and, Fishburn gave similar conditions for partial Pr-representability ($\mathfrak{C}_3$).





- Here is a proof that the 8-proposition fragment above cannot even be *partially* represented by any Pr-function. Note that $\succeq$ contains the following four *strict* judgments:

1. $\mathfrak{s}_1 \succ \mathfrak{s}_2 \vee \mathfrak{s}_4$
2. $\mathfrak{s}_3 \vee \mathfrak{s}_4 \succ \mathfrak{s}_1 \vee \mathfrak{s}_2$
3. $\mathfrak{s}_2 \vee \mathfrak{s}_5 \succ \mathfrak{s}_1 \vee \mathfrak{s}_4$
4. $\mathfrak{s}_1 \vee \mathfrak{s}_2 \vee \mathfrak{s}_4 \succ \mathfrak{s}_3 \vee \mathfrak{s}_5$

- Suppose $\succeq$ *does* have a partial Pr-representation. Then there exists some probability mass function $\mathfrak{m}(\cdot)$ [with five masses $\mathfrak{m}_i = \mathfrak{m}(\mathfrak{s}_i)$] satisfying these four constraints:

(i) $\mathfrak{m}_1 > \mathfrak{m}_2 + \mathfrak{m}_4$

(ii) $\mathfrak{m}_3 + \mathfrak{m}_4 > \mathfrak{m}_1 + \mathfrak{m}_2$

(iii) $\mathfrak{m}_2 + \mathfrak{m}_5 > \mathfrak{m}_1 + \mathfrak{m}_4$

(iv) $\mathfrak{m}_1 + \mathfrak{m}_2 + \mathfrak{m}_4 > \mathfrak{m}_3 + \mathfrak{m}_5$

- But, (i)–(iv) entail that $0 > 0$. Contradiction. *QED.*





- Here is Scott's Axiom, which is an infinite schema.

(SA) Let $\mathbf{X}, \mathbf{Y} \in \prod_m \mathcal{B}$ be (arbitrary) sequences of propositions (from $\mathcal{B}_n$), each having length $m > 0$. Let $\mathbf{X} = \langle x_1, \ldots, x_m \rangle$ and $\mathbf{Y} = \langle y_1, \ldots, y_m \rangle$. *If* conditions (i) and (ii) are satisfied:

(i) $\mathbf{X}$ and $\mathbf{Y}$ *have the same number of truths in every state of* $\mathcal{B}_n$.

(ii) For all $i \in (1, m]$, $x_i \succeq y_i$.

*then*, condition (iii) must also hold

(iii) $y_1 \succeq x_1$.

- Scott shows that {Totality, (A1), $p \succeq \bot$, (SA)} are necessary and sufficient for full Pr-representability of $\succeq$ [*viz.*, ($\mathfrak{C}_4$)].
- Let ($\text{SA}_m$) be the $m$-instance ($m > 0$) of the schema (SA).
- Trivially, (A1) entails ($\text{SA}_1$), *i.e.*, Totality entails Reflexivity.
- It is well known that ($\text{SA}_2$) ⇒ (A5) and ($\text{SA}_3$) ⇒ Transitivity.
- **Q**: What needs to be *super-added* to ($\mathfrak{C}_2$) to ensure ($\mathfrak{C}_4$)?





- **Newsflash**: (A5) ⇒ ($\text{SA}_2$) and (A2) & (A5) ⇒ ($\text{SA}_3$). ∴ (A5) and ($\text{SA}_2$) are *equivalent*, as are (A2) & (A5) and ($\text{SA}_2$) & ($\text{SA}_3$)!
- The Kraft *et. al.* counterexample to ($\mathfrak{C}_3$) involves ($\text{SA}_4$).
- ∴ **A**: The universal claim "$(\forall m \geq 4)(\text{SA}_m)$" is *exactly* what needs to be *super-added* to ($\mathfrak{C}_2$), in order to ensure ($\mathfrak{C}_4$).
- **Fun Fact**: Let ($\text{SA}_n^m$) $\overset{\text{def}}{=}$ the $\langle m, n \rangle$-instance of (SA), where $n$ is the # of states in $\mathcal{B}$. The Kraft *et. al.* counterexample to ($\mathfrak{C}_4$) resides at ($\text{SA}_5^4$). And, this is *smallest in both dimensions*.
- Various complaints about (SA) have been voiced. Fine and others have complained that (SA)'s condition (i) is not a "purely Boolean" condition (it "essentially involves *counting*").
- To be fair, condition (i) of ($\text{SA}_m$) is equivalent to the claim that a specific (antecedently constructible) Boolean formula (with $2m$ variables) is tautological, *i.e.*, that two specific multisets of sets of states of $\mathcal{B}$ are identical. ∴ (SA)'s condition (i) is expressible *via* pure Boolean equations.

- Here is a way to see why (SA$_m$)'s (i) is equivalent (assuming $\mathcal{B}$ is generated by a sentential language $\mathscr{L}$) to the tautologousness of a Boolean $\mathscr{L}$-formula with $2m$ atoms.
- Let's look at the (SA$_2$) case. When $m = 2$, (SA$_m$)'s condition (i) asserts that $\mathbf{X} = \langle x_1, x_2 \rangle$ and $\mathbf{Y} = \langle y_1, y_2 \rangle$ have the same number of truths in every state of $\mathcal{B}$. This means:
  (1) $x_1 \,\&\, x_2 \Dashv\vDash y_1 \,\&\, y_2$,
  (2) $(x_1 \,\&\, \neg x_2) \vee (\neg x_1 \,\&\, x_2) \Dashv\vDash (y_1 \,\&\, \neg y_2) \vee (\neg y_1 \,\&\, y_2)$, and
  (3) $\neg x_1 \,\&\, \neg x_2 \Dashv\vDash \neg y_1 \,\&\, \neg y_2$.
- But, the joint truth of (1)–(3) is equivalent to the logical truth (tautologousness) of the following conjunction:

$$x_1 \,\&\, x_2 \equiv y_1 \,\&\, y_2$$
$$\&$$
$$(x_1 \,\&\, \neg x_2) \vee (\neg x_1 \,\&\, x_2) \equiv (y_1 \,\&\, \neg y_2) \vee (\neg y_1 \,\&\, y_2)$$
$$\&$$
$$\neg x_1 \,\&\, \neg x_2 \equiv \neg y_1 \,\&\, \neg y_2$$





- Here, I will prove that (SA$_2$) entails (A5).
- let $\mathbf{X} = \langle p \vee r, q \rangle$ and $\mathbf{Y} = \langle p \vee q, r \rangle$, where $\langle p, q \rangle$ are mutually exclusive and $\langle p, r \rangle$ are mutually exclusive.
- That is, $x_1 = p \vee r$, $y_1 = p \vee q$, $x_2 = q$, and $y_2 = r$.
- Now, suppose (SA). Then, the ($\Rightarrow$) direction of (A5) follows.
- To see why, assume the left hand side of (A5). That is, suppose that $q \succeq r$, *i.e.*, that $x_2 \succeq y_2$. In the case at hand, this is equivalent to condition (ii) in the antecedent of (SA).
- Thus, in order to establish additivity (A5), all we need to do is show that $(p \vee q) \succeq (p \vee r)$, *i.e.*, that (iii) $y_1 \succeq x_1$.
- This will follow from (SA), provided that we can show condition (i) of (SA) must also be true in this case.
- Indeed, (i) must be true in this case, and this can most easily be seen *via* the following *schematic truth-table*.





| | $p$ | $q$ | $r$ | $\mathfrak{s}_i \vDash p \vee r$? | $\mathfrak{s}_i \vDash q$? | $\mathfrak{s}_i \vDash p \vee q$? | $\mathfrak{s}_i \vDash r$? |
|---|---|---|---|---|---|---|---|
| $\mathfrak{s}_1$ | **T** | **T** | **T** | — | — | — | — |
| $\mathfrak{s}_2$ | **T** | **T** | **F** | — | — | — | — |
| $\mathfrak{s}_3$ | **T** | **F** | **T** | — | — | — | — |
| $\mathfrak{s}_4$ | **T** | **F** | **F** | Yes | No | Yes | No |
| $\mathfrak{s}_5$ | **F** | **T** | **T** | Yes | Yes | Yes | Yes |
| $\mathfrak{s}_6$ | **F** | **T** | **F** | No | Yes | Yes | No |
| $\mathfrak{s}_7$ | **F** | **F** | **T** | Yes | No | No | Yes |
| $\mathfrak{s}_8$ | **F** | **F** | **F** | No | No | No | No |

- Because $\langle p, q \rangle$ and $\langle p, r \rangle$ are mutually exclusive, the (families of) state descriptions $\mathfrak{s}_1$–$\mathfrak{s}_3$ are *impossible*. So, we can ignore those rows of the schematic truth-table.
- Now, in oder to show that (i) holds in this case, we just need to show that each of the five (*possible* families of) state descriptions $\mathfrak{s}_4$–$\mathfrak{s}_8$ satisfies condition (i) of (SA).
- This is easily verified by inspection of the table, since each of these rows contains the same number of "Yes"s in both pairs of columns on the right. □ The ($\Leftarrow$) proof is similar.





- Next, we'll prove *transitivity* of $\succeq$ from (SA$_3$).
- For this proof, we'll need to exploit the fact that (SA) quantifies over (finite) *sequences* of propositions. Let:
  - $\mathbf{X} = \langle x_1, x_2, x_3 \rangle \overset{\text{def}}{=} \langle r, p, q \rangle$.
  - $\mathbf{Y} = \langle y_1, y_2, y_3 \rangle \overset{\text{def}}{=} \langle p, q, r \rangle$.

1. (SA$_3$)
   - Assumption [for $\Rightarrow$I: (SA$_3$) $\Rightarrow$ (A2)].
2. (i) of (SA$_3$).
   - **X** and **Y** *contain the same number of truths in all worlds*, since they involve the same (multiset of) propositions.
3. $p \succeq r \,\&\, q \succeq r$. [*i.e.*, (ii) of (SA$_3$): $x_2 \succeq y_2 \,\&\, x_3 \succeq y_3$]
   - Assumption [for $\Rightarrow$I: $(p \succeq r \,\&\, q \succeq r) \Rightarrow p \succeq r$].
4. $p \succeq r$. [*i.e.*, (iii) of (SA$_3$): $y_1 \succeq x_1$] By 1–3 (logic).
5. $(p \succeq r \,\&\, q \succeq r) \Rightarrow p \succeq r$. [*i.e.*, (A2)] By 3–4 ($\Rightarrow$I).
6. (SA$_3$) $\Rightarrow$ (A2) By 1–5 ($\Rightarrow$I). □