Shortest Axiomatizations of Implicational S4 and S5

Zachary Ernst
Mathematics and Computer Science Division, Argonne National Laboratory, and Philosophy Department, University of Wisconsin–Madison

Branden Fitelson
Philosophy Department, San Jose State University

Kenneth Harris
Mathematics and Computer Science Division, Argonne National Laboratory

Larry Wos
Mathematics and Computer Science Division, Argonne National Laboratory

September, 2002

Abstract. Shortest possible axiomatizations for the implicational fragments of the modal logics S4 and S5 are reported. Among these axiomatizations is included a shortest single axiom for implicational S4—which to our knowledge is the first reported single axiom for that system—and several new shortest single axioms for implicational S5. A variety of automated reasoning strategies were essential to our discoveries.

Keywords: axiomatization, modal logic, implication, single axiom, automated reasoning

1. Background and Conventions

The implicational fragments of the modal logics S4 and S5 have been studied extensively over the years (see, for instance, [8], [7], [1], [5], and [11]). Following tradition, we use the labels "C4" and "C5" to denote the strict implicational fragments of S4 and S5, respectively. Prior [14, Appendix I] reports a variety of Hilbert-style axiomatizations for C4 and C5. All such axiomatizations presuppose condensed detachment as their sole rule of inference (as do ours). We also follow the convention of writing implicational formulas in Polish notation (e.g., instead of the infix "$p \to q$", we use the Polish "Cpq"). When we report our deductions, we use Meredith's D-notation (as explained in Prior's [14, Appendix II]). That is, the notation "$D.a.b$" (appearing to the left of each line in our deductions) is used to denote the most general possible result of detachment (i.e., *condensed* detachment [6]) with a, or some substitution in a, for the major premise $C\alpha\beta$, and with b, or some substitution in b, for the minor premise α. All proofs reported here were discovered with the assistance of the automated reasoning program

OTTER [9]. The extensive role of automated reasoning in the present research is discussed in Section 4.

2. Axiomatic C4

We begin with appropriate background to place the question we answer in perspective.

2.1. A Brief History of Axiomatic C4

The axiomatization of C4 has an interesting history. As far as we can tell, the first time an axiomatization for C4 appeared explicitly in print was in Anderson and Belnap's 1962 paper [1]. Anderson and Belnap report the following 3-axiom basis for C4, which we adopt as our reference C4 axiomatization (the condensed detachment rule, as always, is presupposed to be the sole rule of inference of the systems).

$$\begin{gathered} Cpp \\ CCpqCrCpq \\ CCpCqrCCpqCpr \end{gathered} \tag{1}$$

Anderson and Belnap credit Kripke's 1959 discussion [7] with providing the original insight on how to axiomatize C4. According to Curry [3] and Hacking [5], however, similar work was concurrently being done independently across the Atlantic by Hacking and Smiley. The work of Hacking and Smiley was not published until 1963 [5], but their work on C4 was available in mimeograph form several years before this [3].

Other 3-axiom bases were later discovered for C4 (see [14, Appendix I]), each containing 25 symbols (total) and 11 occurrences of the implication connective C. But, as far as we know, no 2-axiom bases for C4 were ever reported in the literature. Moreover, no single axiom for C4 has been discovered; indeed, this is stated as an open problem in [2, page 83]. Ulrich [16] has shown that C4 is also the strict implicational fragment of each modal logic between S4 and S4.3; hence, our bases are new and shortest bases for the strict implicational fragments of these extensions of S4 as well.

2.2. Shortest Axiomatizations of C4

Using a variety of automated reasoning strategies (see Section 4 for more on these strategies), we have discovered many new 2-axiom bases for C4. The shortest of these include the following 2-basis, which contains only 20 symbols and 9 occurrences of C.

$$CpCqq$$
$$CCpCqrCCpqCsCpr \qquad (2)$$

So far, we have found six such 2-bases, and we know that there are at most eight. We suspect that there exist *exactly* six. We have eliminated all other 2-bases of this complexity except for the following two candidates, whose status remains open: $\{CpCqCrr, CCpqCCqCqrCpr\}$ and $\{CpCqq, CCpqCrCCqCqsCps\}$. We suspect these are *not* bases for C4.[1]

Moreover, we have been able to show that these are the *shortest possible* bases for C4. That is, no other basis for C4 (with any number of axioms) contains fewer symbols (or occurrences of C) than the cited 2-basis. The proof of this result (omitted because of space limitations), which proceeds by exhaustive search of all other possible candidate bases, requires the use of only 20 distinct logical matrices of size ≤ 4. In Section 4, we say a bit more about how this exhaustive search was conducted and how the matrices and bases were discovered.

Our automated reasoning strategies also yielded the following new 21-symbol (10-C) single axiom for C4:

$$CCpCCqCrrCpsCCstCuCpt \qquad (3)$$

As noted earlier, the question of the existence of a single axiom for C4 has been a long-standing open problem in the axiomatics of modal logic [2, page 83]. We have ruled out all shorter single axiom candidates (see Section 4 for more on the strategies used to eliminate and discover single-axiom candidates). Therefore, (3) is a *shortest possible* single axiom for C4. In fact, (3) is *the* shortest C4 single axiom (all other 21-symbol candidates have been eliminated).

With a circle of three deductions, we now establish that each of (2) and (3) is necessary and sufficient for (1). It follows that both (2) and (3) are *bases* for C4. First, we prove (1) ⇒ (3):

	1.	Cpp
	2.	$CCpqCrCpq$

[1] An anonymous referee for this journal points out that this problem can be translated into a problem in combinatory logic because the formulas '$CpCqCrr$ and $CCpqCqCqrCpr$ correspond to the combinators BBK' and $BBB'WB'$ respectively, and the question is whether these suffice to define K', $BBB'K'B'$ and S.'

	3.	$CCpCqrCCpqCpr$
$D.3.3$	4.	$CCCpCqrCpqCCpCqrCpr$
$D.2.3$	5.	$CpCCqCrsCCqrCqs$
$D.3.5$	6.	$CCpCqCrsCpCCqrCqs$
$D.6.2$	7.	$CCpqCCrpCrq$
$D.3.7$	8.	$CCCpqCrpCCpqCrq$
$D.7.2$	9.	$CCpCqrCpCsCqr$
$D.7.8$	10.	$CCpCCqrCsqCpCCqrCsr$
$D.7.9$	11.	$CCpCqCrsCpCqCtCrs$
$D.9.2$	12.	$CCpqCrCsCpq$
$D.12.1$	13.	$CpCqCrr$
$D.9.13$	14.	$CpCqCrCss$
$D.4.13$	15.	$CCpCCqqrCpr$
$D.4.14$	16.	$CCpCCqCrrsCps$
$D.6.16$	17.	$CCpCCqCrrCstCCpsCpt$
$D.15.17$	18.	$CCpCCqCrrCpsCps$
$D.9.18$	19.	$CCpCCqCrrCpsCtCps$
$D.10.19$	20.	$CCpCCqCrrCpsCCstCpt$
$D.11.20$	21.	$CCpCCqCrrCpsCCstCuCpt^*$

Next, we prove that (3) ⇒ (2):

	1.	$CCpCCqCrrCpsCCstCuCpt$
$D.1.1$	2.	$CCCpCqqrCsCCpCCtCuuCpvr$
$D.2.1$	3.	$CpCCqCCrCssCqtCCCuuvCwCqv$
$D.3.3$	4.	$CCpCCqCrrCpsCCCttuCvCpu$
$D.1.3$	5.	$CCCpCqrsCtCCCuurs$
$D.5.1$	6.	$CpCCCqqCrsCCstCuCrt$
$D.6.6$	7.	$CCCppCqrCCrsCtCqs$
$D.4.6$	8.	$CCCppqCrCCstq$
$D.1.6$	9.	$CCCpCqrsCtCCqrs$
$D.1.8$	10.	$CCpqCrCCCsspq$
$D.9.7$	11.	$CpCCqrCCrsCtCqs$
$D.7.10$	12.	$CCCCCppqqrCsCtr$
$D.7.11$	13.	$CCCCpqCrCsqtCuCCspt$
$D.1.12$	14.	$CCpqCrCCCCCssttpq$
$D.12.14$	15.	$CpCqCrCCCCCssttCCuuvv$
$D.15.15$	16.	$CpCqCCCCCrrssCCttuu$
$D.16.16$	17.	$CpCCCCCqqrrCCsstt$
$D.17.17$	18.	$CCCCCppqqCCrrss$
$D.13.18$	19.	$CpCCqCCrrsCqs$

$D.18.12$	20.	$CpCqq^*$
$D.19.19$	21.	$CCpCCqqrCpr$
$D.21.13$	22.	$CCCCpqCrCsqtCCspt$
$D.21.1$	23.	$CCpCCqCrrCpsCtCps$
$D.22.21$	24.	$CCpqCCqrCpr$
$D.24.24$	25.	$CCCCpqCrqsCCrps$
$D.25.25$	26.	$CCpCqrCCsqCpCsr$
$D.24.26$	27.	$CCCCpqCrCpstCCrCqst$
$D.27.21$	28.	$CCCppCqrCCsqCsr$
$D.25.28$	29.	$CCpqCCrpCrq$
$D.28.7$	30.	$CCpCqrCpCsCqr$
$D.24.30$	31.	$CCCpCqCrstCCpCrst$
$D.31.23$	32.	$CCpCpqCrCpq$
$D.29.32$	33.	$CCpCqCqrCpCsCqr$
$D.27.33$	34.	$CCpCqrCCpqCsCpr^*$

Finally, we prove that (2) ⇒ (1), which completes the circle:

	1.	$CpCqq$
	2.	$CCpCqrCCpqCsCpr$
$D.1.1$	3.	Cpp^*
$D.2.2$	4.	$CCCpCqrCpqCsCCpCqrCtCpr$
$D.2.1$	5.	$CCpqCrCpq^*$
$D.4.1$	6.	$CpCCqCqrCsCqr$
$D.6.6$	7.	$CCpCpqCrCpq$
$D.7.7$	8.	$CpCCqCqrCqr$
$D.2.8$	9.	$CCpCqCqrCsCpCqr$
$D.9.2$	10.	$CpCCqCrsCCqrCqs$
$D.10.10$	11.	$CCpCqrCCpqCpr^*$ □

This circle of proofs (1) ⇒ (3) ⇒ (2) ⇒ (1) has the additional property of being *pure*—in the sense of [19] and [20]. That is, (*i*) the proof of (1) ⇒ (3) does not make use of (2), (*ii*) the proof of (3) ⇒ (2) does not make use of (1), and (*iii*) the proof of (2) ⇒ (1) does not make use of (3). We think this circle of pure proofs provides an especially elegant demonstration that (2) and (3) are bases of C4.

3. Axiomatic C5

We begin with appropriate background to place our study of C5 in perspective.

3.1. A Brief History of Axiomatic C5

The problem of axiomatizing the implicational fragment of S5 was solved in 1956 by Lemmon, Meredith, Meredith, Prior, and Thomas. In their seminal paper [8], Lemmon et al. report several bases for C5, including 4-, 3-, 2-, and 1-axiom bases. We adopt the following 3-axiom basis from [8] as our reference axiomatization of C5. (We note that (4) is basis (ii) from Lemmon et al. [8, page 227]. This basis is C. A. Meredith's simplification of Lemmon's original 4-axiom basis for C5; see [11].)

$$\begin{array}{c} CqCpp \\ CCpqCCqrCpr \\ CCCCpqrCpqCpq \end{array} \tag{4}$$

Since the late 1950s, the shortest known bases for C5 have been the 2-axiom bases (v) and (vi) of Lemmon *et al.* [8, page 227]. These bases contain 20 symbols (including 9 occurrences of C). Meredith was able to find the following 21-symbol (10-C) single axiom for C5; and, until now, Meredith and Prior's work [11] seems to have been the last word on this matter.

$$CCCCCppqrCstCCtqCsCsq \tag{5}$$

Especially in view of that success, it is interesting to note that—as far as we know—Meredith *failed* to find a single axiom for C4. This is indeed surprising because Meredith was responsible for finding (shortest) single axioms for almost every system (that has one) that he studied. We sometimes wonder whether the 21-symbol C4 single axiom we reported earlier had been previously discovered (but never published) by Meredith.

3.2. Shortest Axiomatizations of C5

Applying our automated reasoning strategies to C5 (see Section 4), we have discovered several new (and shortest) 2-axiom bases for C5, including the following 18-symbol, 8-C basis.

(6)	Cpp $CCpqCCCCqrsrCpr$

By examining all other possible shorter bases (with any number of axioms), we have established that (6) is a *shortest possible* basis for C5. Furthermore, we have ruled out all other 2-bases of this complexity. Therefore, (6) is *the* shortest basis for C5. A corollary of this result (coupled with the appropriate exhaustive search) asserts that there exists no single axiom for C5 shorter than Meredith's (5). We have, however, discovered the following six other single axioms of length 21:

(7a)	$CCCCpqrCCuuqCCqtCsCpt$
(7b)	$CCCCpqrCCuuqCtCCqsCps$
(7c)	$CCCCpqrCCuutCsCCqtCpt$
(7d)	$CCCCCppqrCuqCCqtCsCut$
(7e)	$CCCCCppqrCuqCCtuCsCtq$
(7f)	$CCCCCppCqrurCCrtCsCqt$

Of these six single axioms, we note that (7d) and (7e) are in the same resonator class as Meredith's previously-known single axiom (5). This means that they differ only with respect to what variables occur in each position; they are identical in each position containing a connective. But in spite of the fact that these three formulas are in the same resonator class, they are not trivial alphabetic variants of each other. We also note that (7d) and (7e) are members of the same resonator class, but are not trivial alphabetic variants.

Due to space constraints, we will not give proofs that each of these six formulas are single axioms for C5. Instead, we will present a circle of three pure proofs (this time using (4) as our reference basis) that together establish that (6) and (7a) are each bases for C5 (and, of course, that their members are each tautologies). First, we prove that (6) ⇒ (4):

	1.	$CCpqCCCCqrsrCpr$
	2.	Cpp
$D.1.1$	3.	$CCCCCCCCpqrqCuqtstCCupt$
$D.1.2$	4.	$CCCCpqrqCpq$
$D.3.3$	5.	$CCpCqrCCuqCpCur$
$D.5.2$	6.	$CCpqCCqrCpr^*$

$D.5.1$	7.	$CCpCCCqrsrCCtqCpCtr$
$D.6.6$	8.	$CCCCpqCrquCCrpu$
$D.6.8$	9.	$CCCCpqrsCCCCqtCptrs$
$D.8.4$	10.	$CCpCqCprCqCpr$
$D.7.9$	11.	$CCpCqrCCCCuqtCurCpCur$
$D.6.10$	12.	$CCCpCqrsCCqCpCqrs$
$D.12.7$	13.	$CCCCpqrCuCCCpqrqCCtpCuCtq$
$D.4.13$	14.	$CCpqCCrpCuCrq$
$D.14.2$	15.	$CCpqCrCpq$
$D.15.2$	16.	$CpCqq^*$
$D.11.16$	17.	$CCCCpqrCpqCuCpq$
$D.6.17$	18.	$CCCpCqrsCCCCqrtCqrs$
$D.17.18$	19.	$CpCCCCqrsCqrCqr$
$D.19.19$	20.	$CCCCpqrCpqCpq^*$

Next, we show that (4) ⇒ (7a).

	1.	$CCpqCCqrCpr$
	2.	$CCCCpqrCpqCpq$
	3.	$CpCqq$
$D.1.1$	4.	$CCCCpqCrquCCrpu$
$D.1.3$	5.	$CCCppqCrq$
$D.4.4$	6.	$CCpCqrCCuqCpCur$
$D.4.2$	7.	$CCpCpqCpq$
$D.6.4$	8.	$CCpCqrCCCCrsCqstCpt$
$D.6.3$	9.	$CCpqCrCpq$
$D.4.7$	10.	$CCCpqpCCpqq$
$D.1.9$	11.	$CCCpCqrsCCqrs$
$D.11.10$	12.	$CCpqCCCpqrr$
$D.6.12$	13.	$CCpCCqrsCCqrCps$
$D.1.5$	14.	$CCCpqrCCCuuqr$
$D.11.13$	15.	$CCCpqrCCpqCur$
$D.1.13$	16.	$CCCCpqCrstCCrCCpqst$
$D.16.2$	17.	$CCpCCCpqrqCpq$
$D.14.17$	18.	$CCCppCCCqrsrCqr$
$D.16.18$	19.	$CCCCpqrCCuuqCpq$
$D.8.19$	20.	$CCCCpqCrquCCCCrptCCsspu$
$D.20.15$	21.	$CCCCpqrCCuuqCCqtCsCpt^*$

Finally, we complete the circle by showing that (7a) ⇒ (6).

	1.	$CCCCpqrCCuuqCCqtCsCpt$
$D.1.1$	2.	$CCCpCqrsCtCCqru$
$D.2.1$	3.	$CpCCCqqrCCrsCtCus$
$D.3.3$	4.	$CCCppqCCqrCuCtr$
$D.1.3$	5.	$CCCCCppqCrCuqtCsCt6t$
$D.2.4$	6.	$CpCCqrCCCqrsCtCsu$
$D.1.5$	7.	$CCCpCqrsCtCCCuuru$
$D.4.6$	8.	$CCCCpqCCCpqrCuCtrsCt6Ct7s$
$D.1.7$	9.	$CCCCCppqqrCuCtr$
$D.1.8$	10.	$CCCCCpqrCuCtrsCt6CCpqs$
$D.1.9$	11.	$CCCpqrCuCCCttCpqr$
$D.1.10$	12.	$CCCCpqCrstCuCCCpqst$
$D.9.11$	13.	$CpCqCrCCCuuCCttss$
$D.12.1$	14.	$CpCCCCqrsrCCrtCuCqt$
$D.13.13$	15.	$CpCqCCCrrCCsstt$
$D.14.14$	16.	$CCCCpqrqCCquCtCpu$
$D.15.15$	17.	$CpCCCqqCCrrss$
$D.16.16$	18.	$CCCCpqCrCuqtCsCCupt$
$D.17.17$	19.	$CCCppCCqqrr$
$D.1.18$	20.	$CCCCpqCprsCtCCqrs$
$D.19.19$	21.	Cpp^*
$D.19.20$	22.	$CCpqCCrpCrq$
$D.9.22$	23.	$CpCqCCrCCsstCrt$
$D.19.23$	24.	$CCpCCqqrCpr$
$D.22.24$	25.	$CCpCqCCrrsCpCqs$
$D.25.16$	26.	$CCCCpqrqCCquCpu$
$D.26.26$	27.	$CCCCpqCrquCCrpu$
$D.24.26$	28.	$CCCCpqrqCpq$
$D.27.27$	29.	$CCpCqrCCuqCpCur$
$D.29.28$	30.	$CCpqCCCCqrsrCpr^*$ □

4. The Role of Automated Reasoning in Our Research

Throughout our investigations into axiomatic C4 and C5, automated reasoning strategies played a crucial role. In particular, we relied heavily on William McCune's automated reasoning program OTTER [9], H. Zhang and J. Zhang's model finder SEM [24], and John Slaney's

model finder MAGIC [15]. Here, we outline the approach used to obtain these results and briefly discuss some of the automated reasoning strategies.

1. First, we wrote computer programs to generate a large list of candidate formulas that were to be tested as axioms. For most of the research, it was practical to generate an exhaustive list of all formulas with as many as twenty-one symbols.

2. All the formulas in the list were tested (by using matrices) to see which were likely to be tautologies in the system in question. Non-tautologies were eliminated from the list of candidate formulas. We used finite matrices rather than decision procedures or semantic arguments because testing for validity on small matrices is very efficient, and those formulas that survived the filters could be subjected to more conclusive tests later in the search.

3. We immediately eliminated large numbers of formulas by applying known results about axiomatizations in the various systems. For example, as reported by Lemmon et al., every axiomatization for C5 must contain a formula with *Cpp* as a (possibly improper) subformula [8]. Another useful result is the Diamond-McKinsey theorem that no Boolean algebra can be axiomatized by formulas containing fewer than three distinct propositional letters [2, p. 83].

4. An arbitrary set of formulas was selected from the list. Using either SEM or a program we ourselves wrote, we found a matrix model that respects modus ponens, invalidates a known axiom basis for the system, but validates the formulas selected from the list. Such a model suffices to show that the formulas are not single axioms for the system.

5. All the remaining formulas in the list were tested against that matrix. Every formula validated by that matrix was eliminated.

6. Steps 4 and 5 were repeated until the list of candidate formulas was reduced to a small number, or eliminated entirely.

7. Finally, we used OTTER to attempt to prove a known axiom basis from each of the remaining candidates. Following the standard approach in automated reasoning, we sought in each case a proof by contradiction and, therefore, assumed the conclusion to be false. By choosing the appropriate list from among those offered by McCune's program, the so-called denial of the conclusion was used only to detect proof completion; any proof that was discovered

relied solely on reasoning forward. Regarding strategy to direct the program's reasoning, we used the resonance strategy, which enables the researcher to provide patterns (formulas or equations) that are treated as attractive because of their functional shape (ignoring their variables) [18]. For the same purpose, we used Veroff's hints strategy in which the researcher provides attractive patterns, patterns that are keyed upon themselves and also on patterns that subsume or are subsumed by them [17]. We restricted the program's reasoning by placing bounds on the complexity of retained conclusions and on the number of distinct letters occurring in such. Purity (of the circles of proofs) was achieved by instructing the program to immediately discard an unwanted specific deduction. With various methodologies based on offered strategies that included the cramming strategy, we also sought proofs of minimal length. With cramming, one instructs the program to rely heavily on chosen steps of a proof with the objective of cramming most or all of them into another proof of interest [21].

Obvious changes were made when we searched for axiom bases with more than a single formula. For example, in contrast to the study of a possible single axiom where its denial was placed in what is called the passive list, the study of a possible basis with more than one member caused us to place its denial in a list called usable. For a second example, when we sought a proof for a basis other than a single axiom, we sometimes used (through the cramming strategy) the proof of one its members to aid us in completing the proof for the entire basis.

Upon implementing the given procedure, we were surprised to discover that even a small number of simple matrix models can eliminate a very large proportion of candidate formulas. For example, by using ten (and possibly fewer) matrices, none of which have more than five elements, one can show that no formula with nineteen symbols is a single axiom for C5. Because of the efficiency of this procedure, we were able to complete all of our searches using a PC and occasionally a Linux workstation.

We believe that this approach for finding axiom bases in Hilbert-style systems could be used for a wide variety of logics, with equal success. For instance, we have used this approach to discover the shortest known basis for the implicational fragment of the logic RM (first axiomatized by Meyer and Parks [12], [13]) [4], and McCune and Veroff have successfully used a similar approach to find short single axioms in lattice theory [10].

Currently, however, an exhaustive search such as the one used in the present study is prohibitively time consuming when applied to

logics with a more complete vocabulary of sentential connectives. The reason rests with the fact that the addition of new connectives causes the number of candidate axiomatizations to increase exponentially. Moreover, when additional connectives are added to the language, the matrices and proofs tend to be larger and more complex. Currently, it is difficult, and sometimes impossible, to discover large matrices for many such problems. Significantly, however, methodologies now exist for using McCune's program to discover extremely complex and difficult proofs. For example, OTTER has yielded proofs consisting of 200 applications of condensed detachment for theorems of significant depth. (See [23] and [22] for information on the solution of challenge problems using OTTER, as well as for open problems.) We believe that further results regarding axiomatizations for more complex logics await future advances in automated reasoning.

Acknowledgments

We thank Michael Byrd and Ted Ulrich for extremely helpful discussions. We would also like to thank an anonymous referee for this journal for helpful comments. This work was supported in part by the Mathematical, Information, and Computational Sciences Division subprogram of the Office of Advanced Scientific Computing Research, U.S. Department of Energy, under Contract W-31-109-Eng-38.

References

1. A. R. Anderson and N. D. Belnap. The pure calculus of entailment. *J. Symbolic Logic*, 27:19–52, 1962.
2. A. R. Anderson and N. D. Belnap. *Entailment*. Princeton University Press, Princeton, N. J., 1975. Volume I: The logic of relevance and necessity.
3. H. B. Curry. Review of Hacking's "What is strict implication?". *Mathematical Reviews*, 31: #4717, 1963.
4. Z. Ernst, B. Fitelson, K. Harris, and L. Wos. A concise axiomatization of $RM_{\rightarrow}$. *Bull. Sect. Logic Univ. Łódź*, 30(4):191–194, 2001.
5. I. Hacking. What is strict implication? *J. Symbolic Logic*, 28:51–71, 1963.
6. J. A. Kalman. Condensed detachment as a rule of inference. *Studia Logica*, 42(4):443–451 (1984), 1983.
7. S. Kripke. The problem of entailment. *J. Symbolic Logic*, 24:324, 1959.
8. E. J. Lemmon, C. A. Meredith, D. Meredith, A. N. Prior, and I. Thomas. *Calculi of pure strict implication*. Philosophy Department, Canterbury University College, Christchurch, New Zealand, 1957.
9. W. McCune. OTTER 3.0 Reference Manual and Guide. Tech. Report ANL-94/6, Argonne National Laboratory, Argonne, IL, 1994.

10. W. McCune, R. Veroff, and R. Padmanabhan. Yet another single law for lattices. Forthcoming in *Algebra Universalis*.
11. C. A. Meredith and A. N. Prior. Investigations into implicational S5. *Z. Math. Logik Grundlagen Math.*, 10:203–220, 1964.
12. R. K. Meyer and R. Z. Parks. Independent axioms for the implicational fragment of Sobociński's three-valued logic. *Zeitschrift für Mathematische Logik Grundlagen Mathematik*, 18:291–295, 1972.
13. R. Z. Parks. A note on R-Mingle and Sobociński's three-valued logic. *Notre Dame Journal of Formal Logic*, XIII(2):227–228, 1972.
14. A. N. Prior. *Formal logic*. Clarendon Press, Oxford, 1962.
15. J. Slaney. MaGIC: Matrix Generator for Implication Connectives. Tech. Report TR-ARP-11-95, Research School of Information Science and Engineer and Centre for Information Science Research, Australian National University, 1995.
16. D. Ulrich. Strict implication in a sequence of extensions of S4. *Z. Math. Logik Grundlag. Math.*, 27(3):201–212, 1981.
17. R. Veroff. Using hints to increase the effectiveness of an automated reasoning program: Case studies. *J. Automated Reason.*, 16(3):223–239, 1996.
18. L. Wos. The resonance strategy. *Computers and Math. with Apps.*, 29(2):133–178, 1995.
19. L. Wos. Searching for circles of pure proofs. *J. Automated Reason.*, 15(3):279–315, 1995.
20. L. Wos. OTTER and the Moufang identity problem. *J. Automated Reason.*, 17(2):215–257, 1996.
21. L. Wos. The strategy of cramming. preprint ANL/MCS-P898-0801, Argonne National Laboratory, Argonne, Illinois, 2001.
22. L. Wos and G. Pieper. *Automated Reasoning and the Discovery of Missing and Elegant Proofs*. Rinton Press. Forthcoming.
23. L. Wos and G. W. Pieper. *A Fascinating Country in the World of Computing: Your Guide to Automated Reasoning*. World Scientific Publishing Company, 2000.
24. J. Zhang and H. Zhang. SEM: A System for Enumerating Models. *Proc. 14th Int. Joint Conf. on AI (IJCAI-95)*, pages 298–303, 1995.