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**Conditionalization Does Not (in general) Maximize Expected Accuracy**

1. **Background**
Rational agents revise their opinions in light of new information they receive. We can think of information-processing as occurring in two stages: exogenous and endogenous.

**The Question**: How should agents revise their beliefs in light of the information they gain *exogenously*?

**Bayesian Answer**: By conditionalization. You conditionalize on E if

\[
p_{new}(\cdot) = p_{old}(\cdot | E)
\]

where

\[
p(A|B) = p(A&B) / p(B).
\]

Since conditionalizing is an operation performed on a *proposition*, thinking of conditionalizing as a way of responding to new information requires thinking that each body of information can be uniquely characterized by a proposition, and that in gaining information one comes to bear some relation to that proposition. I will use the term “exogenously learn” or “learn” for short to describe whatever this relation is.

Why conditionalize? Greaves and Wallace argue for the claim that *conditionalization maximizes expected accuracy*. Their argument for the rationality of conditionalization relies on:

**RatAcc**: The rational update-procedures are those that maximize expected accuracy.

**Thesis 1**: If RatAcc is true, then the rational update-procedure in general is conditionalization*, and not conditionalization.

Conditionalization* has us conditionalize on the proposition *that we learn* P, when P is the proposition we learn.
Thesis 2 (Luminous Infallibility): If RatAcc is true then there exists a (nontrivial) set of propositions that a rational agent is luminously infallible about – that is, a set of propositions that a rational will be certain of if and only if they are true.

2. Formal Framework

Accuracy is measured by a scoring rule, A, which takes a credence function, c, and a state of the world, s and maps the credence-function/state pair to a number between 0 and 1 that represents how accurate the credence function is in that state.

Suppose you know that you’re going to undergo some experience, E. E might be “waking up tomorrow” or “arriving at the office.” Assuming you are probabilistic, for any proposition P, the set \{P, ¬P\} is a partition of your possibility space. So this is a partition of your possibility space:

<table>
<thead>
<tr>
<th>I gain some new information upon undergoing E.</th>
<th>I don’t gain new information upon undergoing E.</th>
</tr>
</thead>
<tbody>
<tr>
<td>I gain (i_1)</td>
<td>I don’t gain new information upon undergoing E.</td>
</tr>
<tr>
<td>I gain (i_2)</td>
<td></td>
</tr>
<tr>
<td>I gain (i_3)</td>
<td></td>
</tr>
<tr>
<td>I gain (i_4)</td>
<td></td>
</tr>
<tr>
<td>...</td>
<td></td>
</tr>
</tbody>
</table>

As are:

<table>
<thead>
<tr>
<th>I learn (X_1)</th>
<th>I don’t gain new information upon undergoing E.</th>
</tr>
</thead>
<tbody>
<tr>
<td>I learn (X_2)</td>
<td></td>
</tr>
<tr>
<td>I learn (X_3)</td>
<td></td>
</tr>
<tr>
<td>I learn (X_4)</td>
<td></td>
</tr>
<tr>
<td>...</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(L(X_1))</th>
<th>(L(X_2))</th>
<th>(L(X_3))</th>
<th>(L(X_4))</th>
<th>(L(T))</th>
</tr>
</thead>
</table>

We’ll call an event in which an agent exogenously learns a proposition a learning experience. An agent who knows she’ll undergo a learning-experience can
represent her future learning-experience by the set of propositions that she assigns non-zero credence to exogenously learning.

An **update-procedure**, $U$, in response to a future learning-experience, $X$, is a function that assigns to each member of $X$, a credence function, with the intended interpretation that an agent performing this update-procedure adopts $U(X_i)$ as her credence function if and only if she learns $X_i$.

Let $A(U(s),s)$ represent the **accuracy score of an agent conforming to $U$ in $s$**.

The **expected accuracy of an update-procedure $U$** in response to a learning-experience $X$, relative to a probability function $p$ is:

$$EA^p(U) = \sum_{s \in L(X)} p(s) A(U(s), s)$$

$$= \sum_{L(X) \in L(X)} \sum_{s \in L(X)} p(s) A(U(X_i)), s)$$

This quantity represents, roughly, how accurate we expect to be as a result of conforming to the update procedure.

### 3. Greaves and Wallace’s Assumptions

**PARTITIONALITY**: The propositions that the agent assigns non-zero credence to exogenously learning form a partition of the agent’s possibility space.

**FACTIVITY**: The agent is certain that if she learns $P$, $P$ is true.

In cases in which PARTITIONALITY and FACTIVITY hold - we will say that the agent’s future learning-experience is representable as “an experiment.”

Is plausible that all rational agents satisfy both of these conditions?

Not obviously: You might think that I could rationally find myself in a position in which I’m certain that I’ll learn exactly one of: $\{P, Q, P&Q\}$, and I assign non-zero credence to each. (In this case PARTITIONALITY fails).
You also might think that I could find myself in the position in which I leave open the possibility of becoming misinformed – that is, I leave open the possibility that the world will “fling” a false proposition into my belief box. (In this case FACTIVITY fails)

4. Three Theorems

\textbf{G&W:} Take any partition of states $\mathcal{P} \{P_1,...,P_n\}$ and consider the set of functions, $\mathcal{F}$, that assign members of $\mathcal{P}$ to probability functions. The member of $\mathcal{F}$, $F$, that maximizes this quantity:

$$\sum_{P_i \in \mathcal{P}} \sum_{s \in P_i} p(s)^* A(F(P_i), s)$$

is: $F(P_i) = \text{Cond} = p(\cdot \mid P_i)$

\textbf{CondMax:} The update-procedure that maximizes expected accuracy relative to a probability function $p$ that satisfies PARTITIONALITY and FACTIVITY is the update-procedure that assigns to each $X_i$ that the agent thinks she might learn: $p(\cdot \mid X_i)$.

\textbf{Generalized CondMax:} The update-procedure that maximizes expected accuracy relative to any probability function $p$ is the update-procedure that assigns to each $X_i$ that the agent thinks she might learn: $p(\cdot \mid L(X_i))$, where $L(X_i)$ is the proposition that the agent learns $X_i$.

5. Three Consequences

RatAcc and Generalized CondMax entail:

\textbf{Cond*:} The rational update-procedure is conditionalization*. In other words, upon learning $P$, an ideally rational agent will conditionize on the proposition that she learned $P$.

Cond* entails:

\textbf{LL:} If one learns $P$, one is rationally required to be certain that one learned $P$. 
**Super Generalized CondMax:** Consider any partition of propositions $P_i$ over a set of states $\Omega$. Let $U$ be a function from $P_i$ to credence functions with the intended interpretation that an agent adopts $U(P_i)$ whenever $P_i$ obtains. The $U$ that maximizes expected accuracy is the one that assigns to each $P_i$ the credence function that results from conditionalizing on $P_i$.

**Luminous Infallibility:** If RatAcc is true then the propositions whose truth determines what credence function it is rational for an agent to adopt are propositions that a rational agent is luminously infallible about – that is, they are propositions that she will be certain of if and only if they are true.