Probability and Graded Truth.
A Qualitative Perspective.

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Graded truth (informal)

Together with a notion of all-or-nothing truth according to which truth is bivalent, there is a notion of truth coming in degrees:

- *Italy is shaped like a boot.*
- *Stanley Kubrick at the end of his life was bald.*
- *The color theme of these slides is blue.*
Graded truth (informal)

Together with a notion of all-or-nothing truth according to which truth is bivalent, there is a notion of truth coming in degrees:

- *Italy is shaped like a boot.*
- *Stanley Kubrick at the end of his life was bald.*
- *The color theme of these slides is blue.*

Graded truth (preformal)

- **qualitative or comparative:** \( \phi \) is more true than \( \psi \).
- **quantitative:** \( \phi \) is true \( x \), typically with \( x \in [0, 1] \)...
Quantitative graded truth and probability

Formal overlap and conceptual confusion in the formalisation of imprecise and uncertain reasoning.
Quantitative graded truth and probability

Formal overlap and conceptual confusion in the formalisation of imprecise and uncertain reasoning.

Outline

Yet again on probability and many-valued logics …
Quantitative graded truth and probability

Formal overlap and conceptual confusion in the formalisation of imprecise and uncertain reasoning.

Outline

Yet again on probability and many-valued logics . . .

. . . with something new

1. a novel argument supporting the distinction between probabilities and degrees of truth,

2. how to bridge the formal and conceptual distinction.
Classical probabilistic logic

Language

- $\mathcal{L} = \{p_1, p_2, \ldots \}$
- $\neg, \rightarrow$
- $SL$
- $\bot$

Definable connectives

- $\theta \lor \phi := \neg \theta \rightarrow \phi$
- $\theta \land \phi := \neg (\neg \theta \lor \neg \phi)$
- $\top := \neg \bot$
Classical probabilistic logic

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- $v: \mathcal{SL} \rightarrow \{0, 1\}$ with truth-tables
- $\models$
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Classical logic

- $\nu: \mathcal{SL} \to \{0, 1\}$ with truth-tables
- $\models$

A probability function over $\mathcal{L}$ is a map $P: \mathcal{SL} \to [0, 1]$ satisfying for all $\theta, \phi \in \mathcal{SL}$

\begin{align*}
(P1) & \text{ if } \models \theta \text{ then } P(\theta) = 1, \\
(P2) & \text{ if } \models \neg (\theta \land \phi) \text{ then } P(\theta \lor \phi) = P(\theta) + P(\phi).
\end{align*}
Real-valued Łukasiewicz logic

\[ v : \mathcal{SL} \to [0, 1] \]

1. \( v(\bot) = 0 \)
2. \( v(\neg \theta) = 1 - v(\theta) \)
3. \( v(\theta \rightarrow \phi) = \begin{cases} 
1, & \text{if } v(\theta) \leq v(\phi); \\
1 - v(\theta) + v(\phi), & \text{otherwise.} 
\end{cases} \)
4. \( v(\theta \vee \phi) = \min\{1, v(\theta) + v(\phi)\} \)
5. \( v(\theta \wedge \phi) = \max\{0, v(\theta) + v(\phi) - 1\} \)
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\[ \models_{\infty} \ (\subset \models) \]
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  5. $v(\theta \land \phi) = \max \{0, v(\theta) + v(\phi) - 1\}$

- $\models_{\infty} (\subset \models)$

For all $\theta, \phi \in \mathcal{S}L$

- (P1*) if $\models_{\infty} \theta$ then $v(\theta) = 1$,
- (P2*) if $\models_{\infty} \neg(\theta \land \phi)$ then $v(\theta \lor \phi) = v(\theta) + v(\phi)$. 
There are occasions, on the other hand, when it seems preferable to start from a purely ordinal relation – i.e. a qualitative one – which either replaces the quantitative notion (should one consider it to be meaningless, or, anyway, if one simply wishes to avoid it), or is used as a first step towards its definition. [...] One could proceed in a similar manner for probabilities, too. (de Finetti, 1935)
There are occasions, on the other hand, when it seems preferable to start from a **purely ordinal relation** – i.e. a qualitative one – which either replaces the quantitative notion (should one consider it to be meaningless, or, anyway, if one simply wishes to avoid it), or is used as a first step towards its definition. [...] One could proceed in a similar manner for probabilities, too. (de Finetti, 1935)

- intuitive appeal
- measurability issues
- axioms as properties
- independence from the numerical apparatus
- more fundamental level
There are occasions, on the other hand, when it seems preferable to start from a purely ordinal relation – i.e. a qualitative one – which either replaces the quantitative notion (should one consider it to be meaningless, or, anyway, if one simply wishes to avoid it), or is used as a first step towards its definition. […] One could proceed in a similar manner for probabilities, too. (de Finetti, 1935)

- intuitive appeal
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- axioms as properties
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- more fundamental level

**Aim:** shedding light on the quantitative side by means of representation theorems
General form of the representation

Qualitative perspective

- comparative judgments
- pairwise evaluation
- $\leq \subseteq X^2$

Quantitative perspective

- numerical assignment
- pointwise evaluation
- $\Phi: X \rightarrow \mathbb{R}$

Necessary and sufficient conditions on a relational structure $\langle X, \leq \rangle$ for the existence of a (n equivalence class of a) real-valued function $\Phi$ such that for all $x, y \in X$

$$x \leq y \iff \Phi(x) \leq \Phi(y).$$
Numerical representability

- **Φ weakly represents** ≥ if
  
  \[ x \geq y \Rightarrow \Phi(x) \geq \Phi(y), \]

- **Φ strongly represents** ≥ if
  
  \[ x \geq y \iff \Phi(x) \geq \Phi(y). \]
Numerical representability

- **Φ weakly represents** \( \succeq \) if
  \[
x \succeq y \implies \Phi(x) \geq \Phi(y),
  \]

- **Φ strongly represents** \( \succeq \) if
  \[
x \succeq y \iff \Phi(x) \geq \Phi(y).
  \]

We are interested in **weak representability**

- Strong representability requires some technical conditions which are not relevant for our discussion and might be misleading.

- Focus: justifying the use of numbers, as values of a measure, starting from plausible properties of the comparative notions.

- The notions are irreducibly comparative.
(\mathcal{A}, S, \emptyset, c, \cup, \cap) is an algebra of events

\succeq \subseteq \mathcal{A}^2 interpreted as being no less probable than

\theta \succ \phi \iff \text{def } \theta \succeq \phi \text{ and not } \phi \succeq \theta \text{ (more probable than)}

\theta \sim \phi \iff \text{def } \theta \preceq \phi \text{ and } \phi \preceq \theta \text{ (as probable as)}
(\mathcal{A}, S, \emptyset, ^c, \cup, \cap) is an algebra of events

\geq \subseteq \mathcal{A}^2 \text{ interpreted as being no less probable than}

\theta \succ \phi \iff \text{def } \theta \geq \phi \text{ and not } \phi \geq \theta \text{ (more probable than)}

\theta \sim \phi \iff \text{def } \theta \leq \phi \text{ and } \phi \leq \theta \text{ (as probable as)}

Definition

A binary relation \geq \subseteq \mathcal{A}^2 is a qualitative probability if it satisfies the following

(QP1) \geq \text{ is total and transitive}

(QP2) A \geq \emptyset, S \succ \emptyset

(QP3) \text{ if } A \cap C = \emptyset, B \cap C = \emptyset \text{ and } A \geq B \text{ then } A \cup C \geq B \cup C
Theorem

If $\succeq \subseteq A^2$ is a qualitative probability and

$$(QP^*) \text{ for each } n \geq 2, \text{ there exists a complete class of } n \text{ incompatible events equally probable,}$$

then there exists a unique function $P: A \to [0, 1]$ such that for all $A, B \in A$

$\begin{align*}
\& P(S) = 1, \\
\& P(A) \geq 0, \\
\& \text{if } A \cap B = \emptyset \text{ then } P(A \cup B) = P(A) + P(B),
\end{align*}$

and

$A \succeq B \Rightarrow P(A) \geq P(B).$
If $\succeq \subseteq \mathcal{S}\mathcal{L}^2$ satisfies

$$(QLP0) \models \theta \Rightarrow \theta \sim \top$$

$$(QLP1) \succeq \text{ is total and transitive}$$

$$(QLP2) \top \succeq \theta, \top > \bot$$

$$(QLP3) \models \neg(\theta \land \chi), \models \neg(\phi \land \chi), \theta \succeq \phi \Rightarrow \theta \lor \chi \succeq \phi \lor \chi$$

$$(QLP\star) \text{ for all } n \geq 2, \text{ there exist } n \text{ events } \theta_1, \ldots, \theta_n \in \mathcal{S}\mathcal{L} \text{ such that }$$

$$(i) \models \bigvee_{i=1}^{n} \theta_i \text{ — collectively exhaustive},$$

$$(ii) \models \neg(\theta_i \land \theta_j) \text{ for } i \neq j \text{ — mutually exclusive},$$

$$(iii) \theta_i \sim \theta_j \text{ for } i \neq j \text{ — equiprobable}.$$ 

then there exists a unique logical probability function $P: \mathcal{S}\mathcal{L} \to [0, 1]$ such that

$$\theta \succeq \phi \Rightarrow P(\theta) \geq P(\phi).$$
The assumption ★

(QLP★) for all \( n \geq 2 \), there exist \( n \) events \( \theta_1, \ldots, \theta_n \in S\mathcal{L} \) such that

(i) \( \models \lor_{i=1}^{n} \theta_i \) — collectively exhaustive,
(ii) \( \models \neg (\theta_i \land \theta_j) \) for \( i \neq j \) — mutually exclusive,
(iii) \( \theta_i \sim \theta_j \) for \( i \neq j \) — equiprobable.

▶ Strong structural assumption (de Finetti, 1931) (Koopman, 1940) (Savage, 1954).

▶ Its intuitive meaning is more compelling than other proposals (Kraft, Pratt & Seidenberg, 1959) (Scott, 1964).

▶ **Set-theoretic perspective**: sequences of tosses of a fair coin

▶ **Logical perspective**: atoms?!

  ▶ infinite partitions can be obtained by allowing for infinitary connectives (Scott & Krauss, 1966)
  ▶ equiprobability of logical valuations?
Qualitative truth (ongoing work)

- $\preceq \subseteq SL$ interpreted as *being no less true than*
Qualitative truth (ongoing work)

$\succeq \subseteq \mathcal{S}\mathcal{L}$ interpreted as *being no less true than*

If $\succeq \subseteq \mathcal{S}\mathcal{L}^2$ satisfies

\begin{align*}
(T0) & \quad \models_{\infty} \theta \Rightarrow \theta \sim \top \\
(T1) & \quad \succeq \text{ is total and transitive} \\
(T2) & \quad \top \succeq \theta, \top \succ \bot \\
(T3) & \quad \theta \succeq \phi \Rightarrow \theta \lor \chi \succeq \phi \lor \chi \\
(T4) & \quad \theta \succeq \phi \Rightarrow \neg \phi \succeq \neg \theta \\
(T5) & \quad (\theta \to \phi) \sim \top \Rightarrow \theta \preceq \phi
\end{align*}

then there exists a unique Łukasiewicz valuation $v: \mathcal{S}\mathcal{L} \to [0,1]$ such that for all $\theta, \phi \in \mathcal{S}\mathcal{L}$

$$\theta \succeq \phi \Rightarrow v(\theta) \geq v(\phi).$$
Comparison of comparisons I

**More or less probable**

(QLP0) \( \models \theta \Rightarrow \theta \sim \top \)

(QLP1) \( \preceq \) is total and transitive

(QLP2) \( \top \preceq \theta, \top \succ \bot \)

(QLP3) \( \models \neg(\theta \land \chi), \models \neg(\phi \land \chi), \theta \preceq \phi \Rightarrow \theta \lor \chi \preceq \phi \lor \chi \)

(QLP\(^\star\)) … uniform partitions …

**More or less true**

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(T5) \( (\theta \rightarrow \phi) \sim \top \Rightarrow \theta \preceq \phi \)
Comparison of comparisons I

More or less probable

(QLP0) $\models \theta \Rightarrow \theta \sim \top$
(QLP1) $\succeq$ is total and transitive
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(QLP3) $\models \neg(\theta \land \chi), \models \neg(\phi \land \chi)$, $\theta \succeq \phi \Rightarrow \theta \lor \chi \succeq \phi \lor \chi$
(QLP*) ...uniform partitions ...

More or less true

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(T4) $\theta \succeq \phi \Rightarrow \neg\phi \succeq \neg\theta$
(T5) $\theta \rightarrow \phi \sim \top \Rightarrow \theta \leq \phi$

Strategy of the proof
Comparison of comparisons II

More or less probable

(QLP0) $\models \theta \Rightarrow \theta \sim T$

(QLP1) $\succeq$ is total and transitive

(QLP2) $T \succeq \theta$, $T \succ \bot$

(QLP3) $\models \neg(\theta \land \chi), \models \neg(\phi \land \chi)$,
\[ \theta \succeq \phi \Rightarrow \theta \lor \chi \succeq \phi \lor \chi \]

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(T5) $(\theta \rightarrow \phi) \sim T \Rightarrow \theta \preceq \phi$

Additivity and normalisation

▶ compositionality

▶ (QLP0) and (T3) are incompatible:

If $\succeq \subseteq \mathcal{SL}^2$ satisfies (T1)–(T5) and $(T0') \models \theta \Rightarrow \theta \sim T$ then there exists a unique classical valuation representing it.

▶ to be or not to be compositional? — (Edgington, 1997) (Bennett, Paris & Vencovská, 2000)
## Interpreting Belief and Truth

<table>
<thead>
<tr>
<th>Interpretation</th>
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<th>Truth</th>
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Qualitative probability can be interpreted subjectively as comparative confidence. Graded truth can be interpreted objectively as graded occurrence. Not only objective/subjective, graded truth and objective chance, objective and agent-independent orderings. The key distinction is to be found elsewhere.
Belief and truth

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## Interpretation

### Belief and truth

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### Not only objective/subjective

- graded truth and objective chance
- objective and agent-independent orderings
- the key distinction is to be found elsewhere
Un evento $E$ può essere:

dal punto di vista logico,

dal punto di vista conoscitivo,

dal punto di vista psicologico (soggettivo) \[ \begin{cases} 
\text{se certo} & \text{Falso} \\
\text{se incerto, con probabilità} & \text{Incerto}
\end{cases} \]

Figura 3.
I tre livelli di conoscenza di un evento.

More or less probable

If $\succeq \subseteq SL^2$ satisfies

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(QLP1) $\succeq$ is total and transitive
(QLP2) $\top \succeq \theta$, $\top \succ \bot$
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(QLP\textdagger) ...uniform partitions ...

then there exists a unique logical probability function $P : SL \rightarrow [0, 1]$ such that

$\theta \succeq \phi \Rightarrow P(\theta) \geq P(\phi)$.

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then there exists a unique Łukasiewicz valuation $v : SL \rightarrow [0, 1]$ such that for all $\theta, \phi \in SL$

$\theta \succeq \phi \Rightarrow v(\theta) \geq v(\phi)$.

Layers: logical indeterminacy and uncertainty
Many-valued or fuzzy events — e.g. (Mundici, 2006)

Plausibility measures (Friedman & Halpern, 1995)

Fuzzy epistemicism (MacFarlane, 2010)

Graded truth as objective probability (ongoing work)
Conclusion

Probability and graded truth from a qualitative perspective:
Probability and graded truth from a qualitative perspective:

- Formal overlapping and conceptual differences between probabilities and degrees of truth are best articulated at a qualitative level of analysis.

- The framework also suggests the conditions under which the distinction can be bridged.
Conclusion

Probability and graded truth from a qualitative perspective:

- Formal overlapping and conceptual differences between probabilities and degrees of truth are best articulated at a qualitative level of analysis.

- The framework also suggests the conditions under which the distinction can be bridged.

Thanks!

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