Probability and Graded Truth. A Qualitative Perspective.

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Motivation I

Graded truth (informal)

Together with a notion of all-or-nothing truth according to which truth is bivalent, there is a notion of truth coming in degrees:

Italy is shaped like a boot.
Stanley Kubrick at the end of his life was bald.
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Graded truth (preformal)

- **qualitative or comparative:** ϕ is more true than ψ.
- **quantitative:** ϕ is true x, typically with $x \in [0, 1]$

Motivation II

Quantitative graded truth and probability

Formal overlap and conceptual confusion in the formalisation of imprecise and uncertain reasoning.

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Outline

Yet again on probability and many-valued logics . . .

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Outline

Yet again on probability and many-valued logics ...

... with something new

1. a novel argument supporting the distinction between probabilities and degrees of truth,
2. how to bridge the formal and conceptual distinction.

Classical probabilistic logic

Language

- $\mathcal{L} = \{p_1, p_2, \dots\}$
- $\neg, \rightarrow$
- $\mathcal{SL}$
- $\bot$

Definable connectives

- $\theta \vee \phi := \neg\theta \rightarrow \phi$
- $\theta \wedge \phi := \neg(\neg\theta \vee \neg\phi)$
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- $\models$

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Classical logic

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- $\models$

A probability function over $\mathcal{L}$ is a map $P \colon \mathcal{SL} \rightarrow [0, 1]$ satisfying for all $\theta, \phi \in \mathcal{SL}$

(P1) if $\models \theta$ then $P(\theta) = 1$,

(P2) if $\models \neg(\theta \wedge \phi)$ then $P(\theta \vee \phi) = P(\theta) + P(\phi)$.

Real-valued Łukasiewicz logic

- $v \colon \mathcal{SL} \to [0,1]$
  1. $v(\bot) = 0$
  2. $v(\neg\theta) = 1 - v(\theta)$
  3. $v(\theta \to \phi) = \begin{cases} 1, & \text{if } v(\theta) \leq v(\phi); \\ 1 - v(\theta) + v(\phi), & \text{otherwise.} \end{cases}$
  4. $v(\theta \vee \phi) = \min\{1, v(\theta) + v(\phi)\}$
  5. $v(\theta \wedge \phi) = \max\{0, v(\theta) + v(\phi) - 1\}$

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For all $\theta, \phi \in \mathcal{SL}$

(P1*) if $\models_\infty \theta$ then $v(\theta) = 1$,

(P2*) if $\models_\infty \neg(\theta \wedge \phi)$ then $v(\theta \vee \phi) = v(\theta) + v(\phi)$.

Qualitative perspective

There are occasions, on the other hand, when it seems preferable to start from a purely ordinal relation – i.e. a qualitative one – which either replaces the quantitative notion (should one consider it to be meaningless, or, anyway, if one simply wishes to avoid it), or is used as a first step towards its definition. [. . .] One could proceed in a similar manner for probabilities, too. (de Finetti, 1935)

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- measurability issues
- axioms as properties
- independence from the numerical apparatus
- more fundamental level

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Aim: shedding light on the quantitative side by means of representation theorems

General form of the representation

Qualitative perspective

- comparative judgments
- pairwise evaluation
- $\preceq \subseteq X^2$

Quantitative perspective

- numerical assignment
- pointwise evaluation
- $\Phi \colon X \to \mathbb{R}$

Necessary and sufficient conditions on a relational structure $\langle X, \preceq \rangle$ for the existence of a(n equivalence class of a) real-valued function Φ such that for all $x, y \in X$

$$x \preceq y \iff \Phi(x) \leq \Phi(y).$$

Numerical representability

- Φ *weakly represents* $\succeq$ if

$$x \succeq y \Rightarrow \Phi(x) \geq \Phi(y),$$

- Φ *strongly represents* $\succeq$ if

$$x \succeq y \Leftrightarrow \Phi(x) \geq \Phi(y).$$

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We are interested in **weak representability**

- Strong representability requires some technical conditions which are not relevant for our discussion and might be misleading.
- Focus: justifying the use of numbers, as values of a measure, starting from plausible properties of the comparative notions.
- The notions are irreducibly comparative.

Qualitative probability (de Finetti, 1931), (Savage, 1954)

- $(\mathcal{A}, S, \emptyset, ^c, \cup, \cap)$ is an algebra of events
- $\succeq \subseteq \mathcal{A}^2$ interpreted as *being no less probable than*
- $\theta \succ \phi \Leftrightarrow_{def} \theta \succeq \phi$ and not $\phi \succeq \theta$ (*more probable than*)
- $\theta \sim \phi \Leftrightarrow_{def} \theta \preceq \phi$ and $\phi \preceq \theta$ (*as probable as*)

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Definition

A binary relation $\succeq \subseteq \mathcal{A}^2$ is a **qualitative probability** if it satisfies the following

(QP1) $\succeq$ is total and transitive

(QP2) $A \succeq \emptyset$, $S \succ \emptyset$

(QP3) if $A \cap C = \emptyset$, $B \cap C = \emptyset$ and $A \succeq B$ then $A \cup C \succeq B \cup C$

Qualitative probability (de Finetti, 1931), (Savage, 1954)

Theorem

If $\succeq \subseteq \mathcal{A}^2$ is a qualitative probability and

(QP$\star$) for each $n \geq 2$, there exists a complete class of n incompatible events equally probable,

then there exists a unique function $P\colon \mathcal{A} \to [0,1]$ such that for all $A, B \in \mathcal{A}$

- $P(S) = 1$,
- $P(A) \geq 0$,
- if $A \cap B = \emptyset$ then $P(A \cup B) = P(A) + P(B)$,

and

$$A \succeq B \Rightarrow P(A) \geq P(B).$$

Qualitative probability: logical reformulation

If $\succeq \subseteq \mathcal{SL}^2$ satisfies

(QLP0) $\models \theta \Rightarrow \theta \sim \top$

(QLP1) $\succeq$ is total and transitive

(QLP2) $\top \succeq \theta$, $\top \succ \bot$

(QLP3) $\models \neg(\theta \wedge \chi), \models \neg(\phi \wedge \chi), \theta \succeq \phi \Rightarrow \theta \vee \chi \succeq \phi \vee \chi$

(QLP$\star$) for all $n \geq 2$, there exist n events $\theta_1, \ldots, \theta_n \in \mathcal{SL}$ such that

(i) $\models \bigvee_{i=1}^{n} \theta_i$ — collectively exhaustive,
(ii) $\models \neg(\theta_i \wedge \theta_j)$ for $i \neq j$ — mutually exclusive,
(iii) $\theta_i \sim \theta_j$ for $i \neq j$ — equiprobable.

then there exists a unique logical probability function $P \colon \mathcal{SL} \to [0, 1]$ such that

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The assumption $\star$

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- Strong structural assumption (de Finetti, 1931) (Koopman, 1940) (Savage, 1954).
- Its intuitive meaning is more compelling than other proposals (Kraft, Pratt & Seidenberg, 1959) (Scott, 1964).

- **Set-theoretic perspective:** sequences of tosses of a fair coin
- **Logical perspective:** atoms?!
  - infinite partitions can be obtained by allowing for infinitary connectives (Scott & Krauss, 1966)
  - equiprobability of logical valuations?

Qualitative truth (ongoing work)

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(T5) $(\theta \to \phi) \sim \top \Rightarrow \theta \preceq \phi$

then there exists a unique Łukasiewicz valuation $v \colon \mathcal{SL} \to [0,1]$ such that for all $\theta, \phi \in \mathcal{SL}$

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Comparison of comparisons I

More or less probable

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(QLP$\star$) ...uniform partitions ...

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Strategy of the proof

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Additivity and normalisation

- compositionality
- (QLP0) and (T3) are incompatible:

> If $\succeq \subseteq \mathcal{SL}^2$ satisfies (T1)–(T5) and (T0$'$) $\models \theta \Rightarrow \theta \sim \top$ then there exists a unique **classical** valuation representing it.

- to be or not to be compositional? — (Edgington, 1997) (Bennett, Paris & Vencovská, 2000)

Interpretation

Belief and truth

	Belief	**Truth**
All-or-nothing	belief, disbelief, suspension of judgment	bivalent truth
Qualitative	more or less probable	more or less true
Quantitative	credences, degrees of belief	degrees of truth

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Not only objective/subjective

- graded truth and objective chance
- objective and agent-independent orderings
- the key distinction is to be found elsewhere

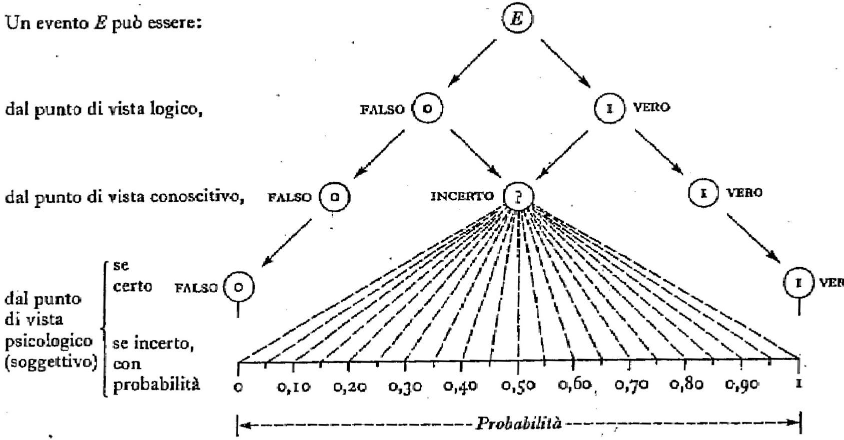

Figura 3.
I tre livelli di conoscenza di un evento.

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Comparison of comparisons III

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Layers: logical indeterminacy and uncertainty

Bridges

- Many-valued or fuzzy events — e.g. (Mundici, 2006)
- Plausibility measures (Friedman & Halpern, 1995)
- Fuzzy epistemicism (MacFarlane, 2010)
- Graded truth as objective probability (ongoing work)

Conclusion

Probability and graded truth from a qualitative perspective:

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- Formal overlapping and conceptual differences between probabilities and degrees of truth are best articulated at a qualitative level of analysis.
- The framework also suggests the conditions under which the distinction can be bridged.

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Thanks!

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