# Single Conclusion Rocks! ... and so does Classical Logic

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# Plan of the talk

- Introduction
- LK, LJ, LM
- Some desiderata from proof-theoretic semantics
- LKS
- Some properties of LKS
- Some desiderata

# Operational meaning: that's why it's considered to rock

*Natural deduction embodies the operational or computational meaning of the logical connectives and quantifiers. The meaning explanations are given in terms of the immediate grounds for asserting a proposition of corresponding form. There can be other, less direct grounds, but these should be reducible to the former for a coherent operational semantics to be possible. The "BHK-conditions" (for Brouwer-Heyting-Kolmogorov), which give the explanations of logical operations of propositional logic in terms of direct provability of propositions, can be put as follows:*

- *A direct proof of the proposition $A\&B$ consists of proofs of the propositions $A$ and $B$.*
- *A direct proof of the proposition $A \vee B$ consists of a proof of the proposition $A$ or a proof of the proposition $B$.*

(Negri/von Plato, 2008, p.5)

A is an object of the form:

$$\Gamma \longrightarrow \Delta$$

$\Gamma, \Delta$: Sequences/lists, multi-sets, sets of formulas

$$A_1, \ldots, A_m \longrightarrow B_1, \ldots, B_n$$

$$A_1 \wedge \cdots \wedge A_m \longrightarrow B_1 \vee \cdots \vee B_n$$

$$\bigwedge \Gamma \longrightarrow \bigvee \Delta$$

- (Ax) $A \longrightarrow A$ ($A$ is prime.)
- *Structural Rules.*

$$\frac{\Gamma, A, B, \Gamma' \longrightarrow \Delta}{\Gamma, B, A, \Gamma' \longrightarrow \Delta}\ (LP) \qquad \frac{\Gamma \longrightarrow \Delta, A, B, \Delta'}{\Gamma \longrightarrow \Delta, B, A, \Delta'}\ (RP)$$

$$\frac{\Gamma \longrightarrow \Delta}{A, \Gamma \longrightarrow \Delta}\ (LW) \qquad \frac{\Gamma \longrightarrow \Delta}{\Gamma \longrightarrow \Delta, A}\ (RW)$$

$$\frac{A, A, \Gamma \longrightarrow \Delta}{A, \Gamma \longrightarrow \Delta}\ (LC) \qquad \frac{\Gamma \longrightarrow \Delta, A, A}{\Gamma \longrightarrow \Delta, A}\ (RC)$$

- Logical rules

$$\frac{\Gamma \longrightarrow \Delta, A}{\neg A, \Gamma \longrightarrow \Delta}\ (L\neg) \qquad \frac{A, \Gamma \longrightarrow \Delta}{\Gamma \longrightarrow \Delta, \neg A}\ (R\neg)$$

$$\frac{A, \Gamma \longrightarrow \Delta}{A \wedge B, \Gamma \longrightarrow \Delta}\ (L\wedge_1) \qquad \frac{\Gamma \longrightarrow \Delta, A \qquad \Gamma \longrightarrow \Delta, B}{\Gamma \longrightarrow \Delta, A \wedge B}\ (R\wedge)$$

$$\frac{A, \Gamma \longrightarrow \Delta \qquad B, \Gamma \longrightarrow \Delta}{A \vee B, \Gamma \longrightarrow \Delta}\ (L\vee) \qquad \frac{\Gamma \longrightarrow \Delta, A}{\Gamma \longrightarrow \Delta, A \vee B}\ (R\vee_1)$$

$$\frac{\Gamma \longrightarrow \Delta, A \qquad B, \Gamma \longrightarrow \Delta}{A \rightarrow B, \Gamma \longrightarrow \Delta}\ (L \rightarrow) \qquad \frac{A, \Gamma \longrightarrow \Delta, B}{\Gamma \longrightarrow \Delta, A \rightarrow B}\ (R \rightarrow)$$

# LK cont'd

$$\frac{A[a], \Gamma \longrightarrow \Delta}{\forall x A[x], \Gamma \longrightarrow \Delta} \ (L\forall) \qquad \frac{\Gamma \longrightarrow \Delta, A[a]}{\Gamma \longrightarrow \Delta, \forall x A[x]} \ (R\forall)^*$$

$$\frac{A[a], \Gamma \longrightarrow \Delta}{\exists x A[x], \Gamma \longrightarrow \Delta} \ (L\exists)^* \qquad \frac{\Gamma \longrightarrow \Delta, A[a]}{\Gamma \longrightarrow \Delta, \exists x A[x]} \ (R\exists)$$

$^*$ $a$ does not occur below the inference line.

$A \longrightarrow A$ for all $A$.

- *Cut*

$$\frac{\Gamma \longrightarrow \Delta, F \quad F, \Gamma' \longrightarrow \Delta'}{\Gamma, \Gamma' \longrightarrow \Delta, \Delta'} \; Cut$$

- *Multicut/Mix*

$$(\text{Multicut, Mix}) \; \frac{\Gamma \longrightarrow \Delta, F^n \qquad F^m, \Phi \longrightarrow \Psi}{\Gamma, \Gamma' \longrightarrow \Delta, \Delta'} \; (n, m > 0)$$

**Theorem**

*Cut/Mix is eliminable.*

# Sub-formula property

## Definition (Sub-formula)

(a) $A$ is atomic: $A$ is a subformula of $A$.

(b) $A$ is $\neg B$: $\neg B$ and all subformulas of $B$ are subformulas of $A$.

(c) $A$ is $B \dagger C$: $B \dagger C$, $B, C$ and all subformulas of $B$ and $C$ are subformulas of $A$.

(d) $A$ is $\forall x B[x]$: $\forall x B[x]$ and $B[b]$ for all $b$ are subformulas of $A$.

(e) Subformulas of a sequent are all subformulas of formulas of it.

## Theorem (Subformula property)

*All formulas in the cut-free derivation of $\Gamma \longrightarrow \Delta$ in LK are subformulas of $\Gamma, \Delta$.*

## Definition (Consistency)

LK is consistent off ... $\longrightarrow$ ... (i.e. the empty sequent) is not provable in LK.

## Theorem (Consistency of LK)

*LK is consistent.*

# LK & LM

- LJ comes from restricting LK to at most one formula on the right.
- LM comes from restricting LK to exactly one formula to the right plus dropping $(L\neg)$, and $(R\neg)$ plus defining: $\neg A := A \to \bot$, and $\bot \longrightarrow \bot$ is also an axiom.

# Some desiderata from proof-theoretic semantics

- **Weakly explicit**: the introduction rules for a logical operator ‡ are *weakly explicit* iff ‡ occurs only in the lower sequent.
- **Explicit**: weakly explicit + ‡ occurs only once on the left or on the right side of the sequent.
- **Separeted**: The introduction rules for a logical operator ‡ are *separated* iff no other logical operator occurs in the introduction rules.
- **Weakly symmetric**: the introduction rules for a logical constant ‡ are *weakly symmetric* iff every rule belongs either to the left or to the right introduction rules.
- **Symmetric**: weakly symmetric + neither the right side of a left intro-rule, nor the left side of a right intro-rule is empty.

- **Uniqueness** - basically: If two logical operators ‡ and † are governed by the same inference rules, then both $\dagger\Gamma \longrightarrow \ddagger\Delta$ and $\ddagger\Gamma \longrightarrow \dagger\Delta$ are derivable by use of the rules for ‡, †, and the axioms only.
- **Conservativeness** - baiscally: If a logical operator ‡ is added to a sequent calculus, then every proof of an ‡-free sequent must be convertible into a proof of this sequent without the use of a ‡-rule.
- **Derivability of Identicals** - basically: The logical constants must be uniquely determined by their inferences rules, i.e. for a given logical operator ‡, for all $A_1, \ldots, A_n$: the sequent $\dagger(A_1, \ldots, A_n) \longrightarrow \dagger(A_1, \ldots, A_n)$ must be derivable by use of the rules of † and the axiom only.
- **Transitivity** - Cut-elimination, analyticity
- **Dilution/Weakening**

# LKS

- (Ax) $A \longrightarrow A$ ($A$ is prime.)
- *Structural Rules*.

$$\frac{\Gamma, A, B, \Gamma' \longrightarrow C}{\Gamma, B, A, \Gamma' \longrightarrow C}\ (LP)$$

$$\frac{\Gamma \longrightarrow C}{A, \Gamma \longrightarrow C}\ (LW) \qquad \frac{\Gamma \longrightarrow}{\Gamma \longrightarrow A}\ (RW)$$

$$\frac{A, A, \Gamma \longrightarrow C}{A, \Gamma \longrightarrow C}\ (LC)$$

# LKS cont'd

- Logical rules

$$\frac{\Gamma \longrightarrow A}{\neg A, \Gamma \longrightarrow} \ (L\neg) \qquad \frac{A, \Gamma \longrightarrow}{\Gamma \longrightarrow \neg A} \ (R\neg)$$

$$\frac{A, \Gamma \longrightarrow C}{\neg\neg A, \Gamma \longrightarrow C} \ (L\neg\neg) \qquad \frac{\Gamma \longrightarrow A}{\Gamma \longrightarrow \neg\neg A} \ (R\neg\neg)$$

$$\frac{A, \Gamma \longrightarrow C}{A \wedge B, \Gamma \longrightarrow C} \ (L\wedge_1) \qquad \frac{\Gamma \longrightarrow A \qquad \Gamma \longrightarrow B}{\Gamma \longrightarrow A \wedge B} \ (R\wedge)$$

$$\frac{A, \Gamma \longrightarrow C \qquad B, \Gamma \longrightarrow C}{A \vee B, \Gamma \longrightarrow C} \ (L\vee) \qquad \frac{\Gamma \longrightarrow A}{\Gamma \longrightarrow A \vee B} \ (R\vee_1)$$

$$\frac{\Gamma \longrightarrow A \qquad B, \Gamma \longrightarrow C}{A \rightarrow B, \Gamma \longrightarrow C} \ (L \rightarrow) \qquad \frac{A, \Gamma \longrightarrow B}{\Gamma \longrightarrow A \rightarrow B} \ (R \rightarrow)$$

- *Cut*

$$\frac{\Gamma \longrightarrow F \quad F, \Gamma' \longrightarrow C}{\Gamma, \Gamma' \longrightarrow C}\ \mathit{Cut}$$

- *Multicut/Mix*

$$(\text{Multicut, Mix})\ \frac{\Gamma \longrightarrow F \qquad F^m, \Gamma' \longrightarrow C}{\Gamma, \Gamma' \longrightarrow C}\ (m > 0)$$

# Some results

### Conjecture (Cut/Mix)

*LKS enjoys cut-elimination.*

### Definition (Sub-formula)

(a) $A$ is atomic: $A$ is a subformula of $A$.

(b) $A$ is $\neg B$: $\neg B$ and all subformulas of $B$ are subformulas of $A$.

(c) $A$ is $B \dagger C$: $B \dagger C$, $B, C$ and all subformulas of $B$ and $C$ are subformulas of $A$.

(d) $A$ is $\forall x B[x]$: $\forall x B[x]$ and $B[b]$ for all $b$ are subformulas of $A$.

(e) Subformulas of a sequent are all subformulas of formulas of it.

### Conjecture (Subformula property)

*All formulas in the cut-free derivation of $\Gamma \longrightarrow \Delta$ in LKS are subformulas of $\Gamma$, $\Delta$.*

**Definition (Consistency)**

LKS is consistent iff ... $\longrightarrow$ ... (i.e. the empty sequent) is not provable in LKS.

**Conjecture (Consistency of LKS)**

*LKS is consistent.*

The following rule:

$$\frac{A, \Gamma \longrightarrow C \qquad \neg A, \Gamma \longrightarrow C}{\Gamma \longrightarrow C} \text{(Gem)}$$ [1]

is derivable in LKSp.

Proof.

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{A, \Gamma \longrightarrow C}{\neg C, A, \Gamma \longrightarrow}}{\neg C, \Gamma \longrightarrow \neg A} \qquad \neg A, \Gamma \longrightarrow C}{\neg C, \Gamma \longrightarrow C}}{\neg C, \neg C, \Gamma \longrightarrow}}{\neg C, \Gamma \longrightarrow}}{\Gamma \longrightarrow \neg\neg C}} \qquad \neg\neg C \longrightarrow C}{\Gamma \longrightarrow C}$$

## Fact

*Gem allows for a proof of* $\longrightarrow A \vee \neg A$.

[1] Negri/von Plato, 2008, p.113f.

# LKS cont'd

$$\frac{A[a], \Gamma \longrightarrow C}{\forall x A[x], \Gamma \longrightarrow C}\ (L\forall) \qquad \frac{\Gamma \longrightarrow A[a]}{\Gamma \longrightarrow \forall x A[x]}\ (R\forall)^*$$

$$\frac{A[a], \Gamma \longrightarrow C}{\exists x A[x], \Gamma \longrightarrow C}\ (L\exists)^* \qquad \frac{\Gamma \longrightarrow A[a]}{\Gamma \longrightarrow \exists x A[x]}\ (R\exists)$$

$$\frac{A[a], \Gamma \longrightarrow C}{\neg\forall x \neg A[x], \Gamma \longrightarrow C}\ (L\neg\forall\neg)^* \qquad \frac{\Gamma \longrightarrow A[a]}{\Gamma \longrightarrow \neg\forall x \neg A[x]}\ (R\neg\forall\neg)$$

$$\frac{A[a], \Gamma \longrightarrow C}{\neg\exists x \neg A[x], \Gamma \longrightarrow C}\ (L\neg\exists\neg) \qquad \frac{\Gamma \longrightarrow A[a]}{\Gamma \longrightarrow \neg\exists x \neg A[x]}\ (R\neg\exists\neg)^*$$

$^*$ $a$ does not occur below the inference line.

$A \longrightarrow A$ for all $A$.

**Conjecture**

*Cut, sub-formula, consistency all hold for full LKS.*

# LKS - an admissible rule

$$\frac{A, \Gamma \longrightarrow C \qquad B, \Gamma \longrightarrow D}{A \vee B, \Gamma \longrightarrow C \vee D}$$

# LKS - cont'd

**Theorem**

*If LKS* $\vdash \Gamma \longrightarrow A \vee B$*, then LKS* $\vdash \Gamma \longrightarrow A$ *or LKS* $\vdash \Gamma \longrightarrow B$*.*

Proof by induction on the height of a derivation.

Given this fact the following rules are admissible:

$$\frac{\Gamma \longrightarrow A \vee B[a]}{\Gamma \longrightarrow A \vee \forall x B[x]} \text{ (Condition on the eigenvariable!)}$$

$$\frac{\Gamma \longrightarrow A \vee B[a]}{\Gamma \longrightarrow A \vee \neg\forall x \neg B[x]}$$

Likewise for $\exists$ and $\neg\exists\neg$.

**Theorem**

*If LK* $\vdash \Gamma \longrightarrow \Delta$*, then LKS* $\vdash \Gamma \longrightarrow \bigvee \Delta$*.*

**Theorem**

*If LKS* $\vdash \Gamma \longrightarrow \bigvee \Delta$*, then LK* $\vdash \Gamma \longrightarrow \Delta$*.*

Both theorems a established by proofs on the height of a derivation.
For the second theorem: Suppose, as an example: LKS $\vdash^n \Gamma \longrightarrow \bigvee \Delta \vee A \vee B$, with last premiss is LKS $\vdash^{n-1} \Gamma \longrightarrow \bigvee \Delta \vee A$. So by i.h. LK $\vdash \bigvee \Delta$ and also LK $\vdash \bigvee \Delta \vee A \longrightarrow \Delta, A$; so by an application of Cut on $\bigvee \Delta \vee A$, this sequent is obtained: LK $\vdash \Gamma \longrightarrow \Delta, A$ with an application of $(R\vee_1)$ the required conclusion follows.

**Thank you.**

**Blocks of a formula**

- $\mathrm{b}(A) = 0$, if $A$ is prime.
- $\mathrm{b}(A \circ B) = \mathrm{b}(A) + \mathrm{b}(B) = 1$
- $\mathrm{b}(\neg A) = \mathrm{b}(\neg\neg A) = \mathrm{b}(A) + 1$
- $\mathrm{b}(\exists x A[x]) = \mathrm{b}(\neg \exists x \neg A[x]) = \mathrm{b}(A) + 1$
- $\mathrm{b}(\forall x A[x]) = \mathrm{b}(\neg \forall x \neg A[x]) = \mathrm{b}(A) + 1$

$\langle w, \mathrm{b}, h \rangle < \langle w', \mathrm{b}', h' \rangle$

iff

$(w < w')$ or $(w = w'$ and $\mathrm{b} < \mathrm{b}')$ or $(w = w'$ and $\mathrm{b} = \mathrm{b}'$ and $h < h')$.

# A Problem

$$\dfrac{\dfrac{\neg A, \Gamma \longrightarrow}{\Gamma \longrightarrow \neg\neg A} \qquad \dfrac{A, \Gamma' \longrightarrow B}{\neg\neg A, \Gamma' \longrightarrow B}}{\Gamma, \Gamma' \longrightarrow B}$$

A proof-transformation works only if the following inversion rule is hp-admissible:

If LKS $\vdash^n \neg\neg A, \Gamma \longrightarrow C$, then LKS $\vdash^n A, \Gamma \longrightarrow C$.