Learning in the Limit, General Topology and Modal Logic

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Bridges 2
Rutgers University, September 19th, 2015

Outline

Introduction

Characterization of Learnability and Solvability

Constructive Order-driven Learning

Towards Epistemic Logic of Learnability

Intermediate (but Interesting) Conclusions

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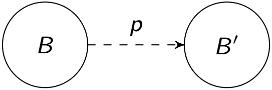
Background

- Learning and belief revision go their separate ways,
- conjecture dynamics is a common theme.
- What are the principles of this dynamics?



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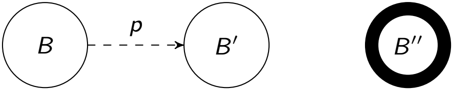

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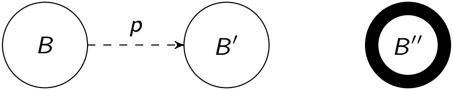
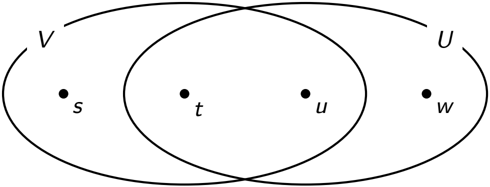


Truth-tracking!

DEFINITION

An *epistemic space* is a pair $\mathbb{S} = (S, \mathcal{O})$ consisting of a state space (a set of possible worlds) S and a countable set of observable properties $\mathcal{O} \subseteq \mathcal{P}(S)$.



Definition

Let $\mathbb{S} = (S, \mathcal{O})$ be an epistemic space.

- A *data stream* is an infinite sequence $\vec{O} = (O_0, O_1, \ldots)$ of data from $\mathcal{O}$.
- A *data sequence* is a finite sequence $\sigma = (\sigma_0, \ldots, \sigma_n)$.

Definition

Take $\mathbb{S} = (S, \mathcal{O})$ and $s \in S$. A data stream $\vec{O}$ is:

- *sound with respect to s* iff every element listed in $\vec{O}$ is true in s.
- *complete with respect to s* iff every observable true in s is listed in $\vec{O}$.

We assume that data streams are sound and complete.

Definition

Let $\mathbb{S} = (S, \mathcal{O})$ be an epistemic space and let $\sigma_0, \ldots, \sigma_n \in \mathcal{O}$. A *learner* is a function L that on the input of $\mathbb{S}$ and data sequence $(\sigma_0, \ldots, \sigma_n)$ outputs some set of worlds $L(\mathbb{S}, (\sigma_0, \ldots, \sigma_n)) \subseteq S$, called a *conjecture*.

Definition

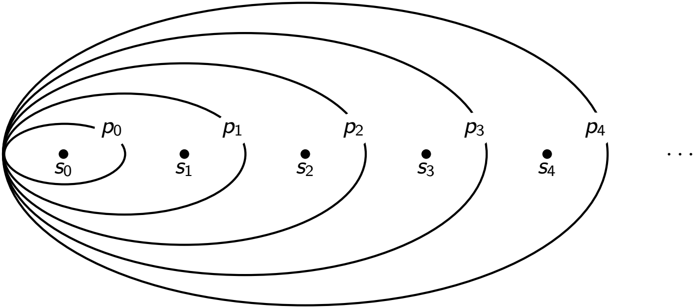
$\mathbb{S} = (S, \mathcal{O})$ is *learnable by* L if for every state $s \in S$ we have that for every sound and complete data stream $\vec{O}$ for s, there is $n \in \mathbb{N}$ s.t.:

$$L(\mathbb{S}, (O_0, \ldots, O_k)) = \{s\} \text{ for all } k \geq n.$$

An epistemic space $\mathbb{S}$ is *learnable* if it is learnable by a learner L.

Example of a Learnable Space

Let $\mathbb{S} = (S, \mathcal{O})$ such that $S = \{s_n \mid n \in \mathbb{N}\}$, $\mathcal{O} = \{p_i \mid i \in \mathbb{N}\}$, and for any $k \in \mathbb{N}$, $p_k = \{s_i \mid 0 \leq i \leq k\}$. $\mathbb{S}$ is learnable.



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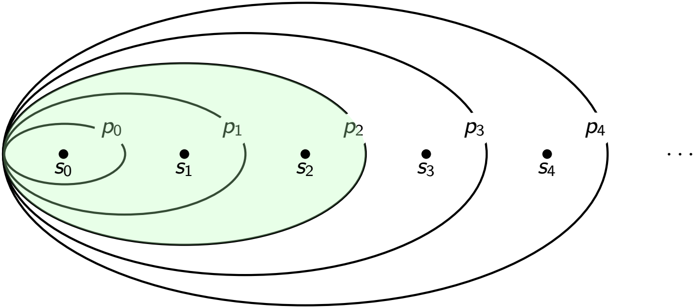

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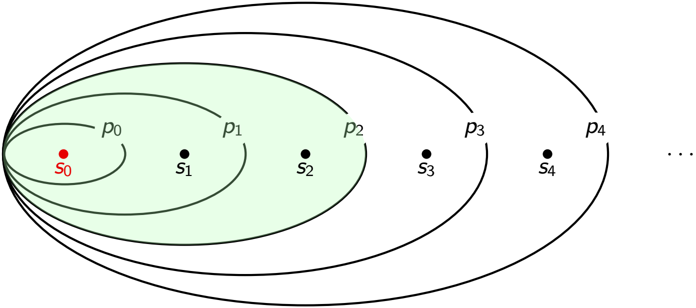

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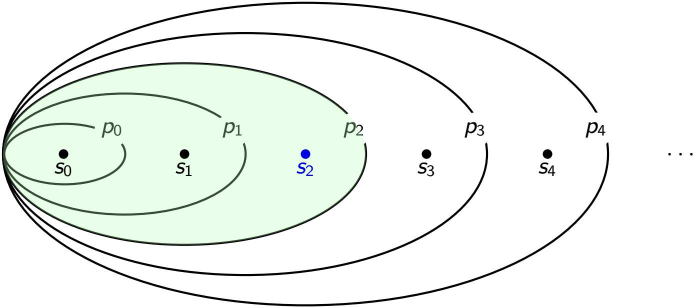
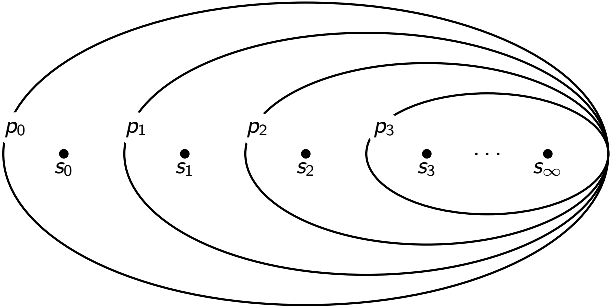


Example of a Non-Learnable Space

Consider $\mathbb{S} = (S, \mathcal{O})$, where $S := \{s_n \mid n \in \mathbb{N}\} \cup \{s_\infty\}$, and $\mathcal{O} = \{p_i \mid i \in \mathbb{N}\}$, and for any $k \in \mathbb{N}$, $p_k := \{s_k, s_{k+1}, \ldots\} \cup \{s_\infty\}$. $\mathbb{S}$ is not learnable.



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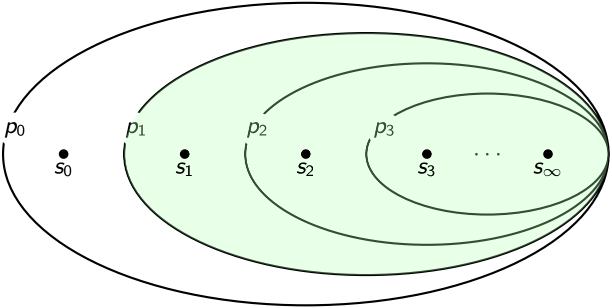

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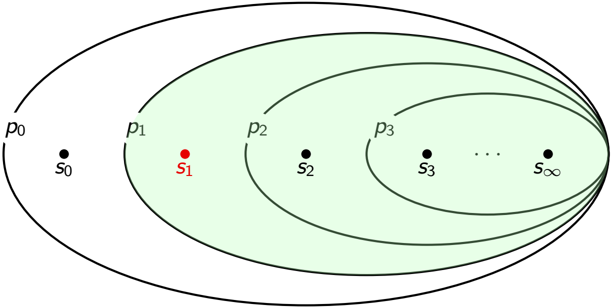

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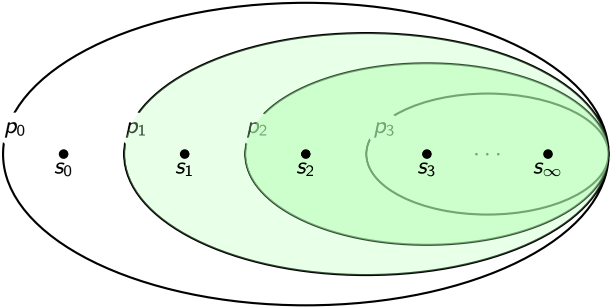

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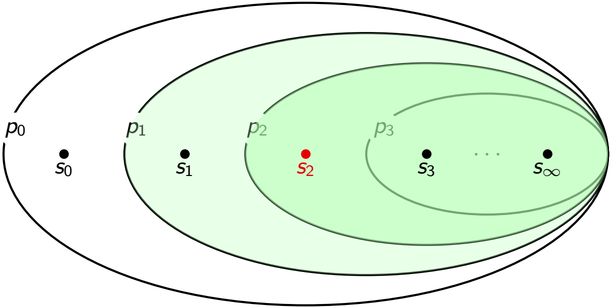


Questions, Answers, and Problems

Definition

A *question* $\mathcal{Q}$ is a partition of S, whose cells A_i are called *answers to* $\mathcal{Q}$.
Given $s \in A \subseteq S$, $A \in \mathcal{Q}$ is called *the answer to* $\mathcal{Q}$ *at* s, denoted A_s.

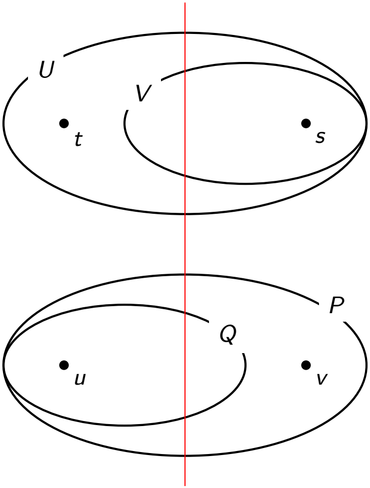
Definition

$\mathcal{Q}'$ is a *refinement* of $\mathcal{Q}$ if all answers of $\mathcal{Q}$ is a disjoint union of answers of $\mathcal{Q}'$.

Definition

A *problem* $\mathbb{P}$ is a pair $(\mathbb{S}, \mathcal{Q})$ consisting of $\mathbb{S} = (S, \mathcal{O})$ and $\mathcal{Q}$ over S.
$\mathbb{P}' = (\mathbb{S}, \mathcal{Q}')$ is a *refinement* of $\mathbb{P} = (\mathbb{S}, \mathcal{Q})$ if $\mathcal{Q}'$ is a refinement of $\mathcal{Q}$.

Illustration



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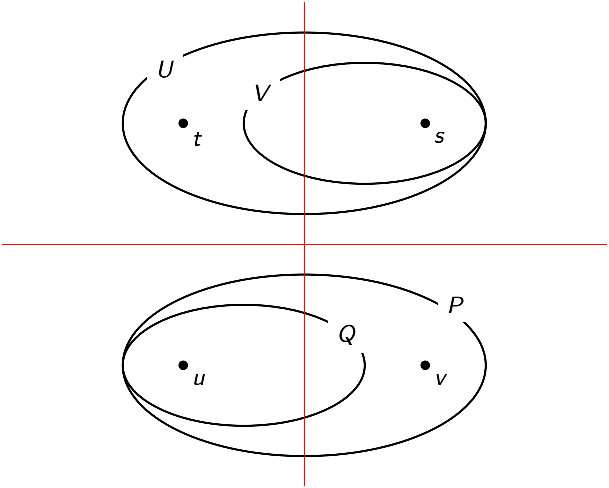


Solving in the Limit

Definition

A learning method L *solves a problem* $\mathbb{P} = (\mathbb{S}, \mathcal{Q})$ in the limit iff for every state $s \in S$ and every data stream $\vec{O}$ for s, there exists some $k \in \mathbb{N}$ such that:

$$L(\mathbb{S}, \vec{O}[n]) \subseteq A_s \text{ for all } n \geq k.$$

A problem is *solvable in the limit* if there is a learner that solves it in the limit.

General Topology

Definition

A topology τ over a set S is a collection of subsets of S (open sets) s.t.:

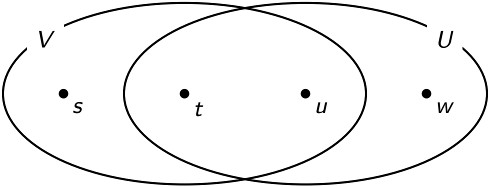
1. $\emptyset \in \tau$,
2. $S \in \tau$,
3. for any $X \subseteq \tau$, $\bigcup X \in \tau$, and
4. for any finite $X \subseteq \tau$ we have $\bigcap X \in \tau$.

Definition

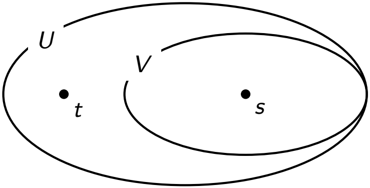
Take a set $X \subseteq S$.

1. The *interior* of X: $Int(X) = \bigcup\{U \in \tau \mid U \subseteq X\}$.
2. A subset $Y \subseteq S$ is *closed* if an only if its complement, Y^c is open.
3. The *closure* of X: $\overline{X} = (Int(X^c))^c = \bigcap\{Y \mid X \subseteq Y \text{ and } Y \text{ is closed}\}$.

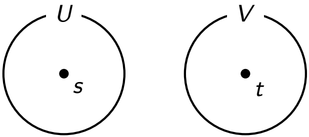
Separability by observations: Illustration



(A) t and u are not separable



(B) weakly separated space, $T0$



(C) strongly separated space, $T1$

DEFINITION

A topology τ is T_d iff for every $s \in S$ there is a $U \in \tau$ such that $U \setminus \{s\} \in \tau$, i.e., for every $s \in S$ there is a $U \in \tau$ such that $\{s\} = U \cap \overline{\{s\}}$.
T_d is a separation property between $T0$ and $T1$.

DEFINITION

A set A is *locally closed* if $A = U \cap C$, where U is open and C is closed.

A set is *constructible* if it is a finite disjoint union of locally closet sets.

An ω-*constructible set* is a countable union of locally closed sets.

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The Topology Associated with an Epistemic Space

Definition

The topology $\tau_{\mathbb{S}}$ associated with an epistemic space $\mathbb{S} = (S, \mathcal{O})$ is a collection of subsets of S of the following properties:

1. for any $O \in \mathcal{O}$ it is the case that $O \in \tau_{\mathbb{S}}$
2. $\emptyset \in \tau_{\mathbb{S}}$,
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4. for any $U \subseteq \tau_{\mathbb{S}}$, $\bigcup U \in \tau_{\mathbb{S}}$, and
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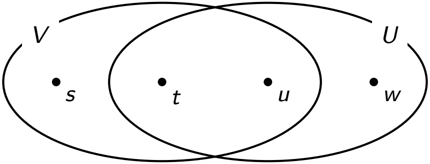

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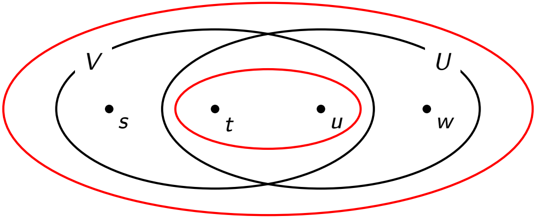

Characterization of Solvability in the Limit

Theorem

A problem $\mathbb{P} = (\mathbb{S}, \mathcal{Q})$ is solvable in the limit iff $\mathcal{Q}$ has a locally closed refinement.

Corollary

An epistemic space $\mathbb{S} = (S, \mathcal{O})$ is learnable in the limit iff it satisfies the T_d separation axiom.

Outline

Order-driven learning: Motivation

- Belief Revision: minimal states give beliefs.
- Computational Learning Theory: co-learning, learning by erasing.
- Philosophy of Science: Ockham's razor.

Conditioning

Definition

Conditioning wrt a prior $\leq$ on S, is defined in the following way:

$$L_{\leq}(O_1, \ldots, O_n) := Min_{\leq}\left(\bigcap_{i=1}^{n} O_i\right)$$

whenever $\bigcap_i O_i$ has any minimal elements; and otherwise:

$$L_{\leq}(O_1, \ldots, O_n) := \bigcap_{i=1}^{n} O_i.$$

Definition

Conditioning is said to be *standard* if the prior $\leq$ is *well-founded*.

Theorem

Non-standard conditioning is a universal problem solving method.

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Logic for Learnability

Since learnability is about potentially successful changes of **beliefs** one expects some doxastic logic to capture it and to reason about it.

Definition (Syntax)

Take countable set of propositional symbols P.

$$\varphi := p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box\varphi,$$

for all $p \in P$, the usual abbreviations are $\vee$, $\rightarrow$, and $\Diamond$.

Relational semantics for modal logic

Definition (Syntax)

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for all $p \in P$, the usual abbreviations are $\vee$, $\rightarrow$, and $\Diamond$.

Definition (Semantics)

Given a model $M = (W, R, v)$, where $v : P \rightarrow \wp(W)$, and a state $x \in W$:

$M, x \models p$	iff	$x \in v(p)$ for each $p \in P$
$M, x \models \neg\varphi$	iff	not $M, x \models \varphi$
$M, x \models \varphi \wedge \psi$	iff	$M, x \models \varphi$ and $M, x \models \psi$
$M, x \models \Box\varphi$	iff	for all $y \in W$: if xRy then $M, y \models \varphi$
and dually:		
$M, x \models \Diamond\varphi$	iff	there is $y \in W$: xRy and $M, y \models \varphi$

Some Axioms and Their Epistemic Meaning

Rules

(MP) if $\vdash \varphi$ and $\vdash \varphi \to \psi$, then $\vdash \psi$

(N) if $\vdash \varphi$, then $\vdash \Box\varphi$

Axioms

(K) $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ (omniscience)

(T) $\Box\varphi \to \varphi$ (truthfullness/reflexivity)

(D) $\Box\varphi \to \neg\Box\neg\varphi$ (consistency/seriality)

(4) $\Box\varphi \to \Box\Box\varphi$ (positive introspection/transitivity)

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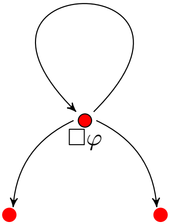
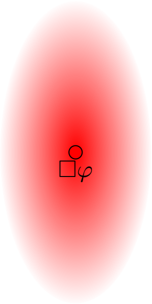
Ax is a logic of a class of models $\mathcal{M}$ iff *Ax* is sound and complete wrt $\mathcal{M}$.

Can we use modal logic on topologies?

Relational $\Box$ vs Topological $\Box := \mathit{Int}$





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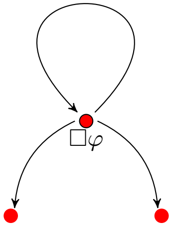

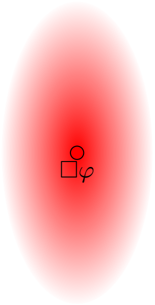

Definition

Let P be a set of propositional symbols. A topological model (or a topo-model) $M = (X, \mathcal{O}, v)$ is a topological space $\tau = (X, \mathcal{O})$ together with a valuation function $v : P \to \wp(X)$.

Definition

Truth of modal formulas is defined inductively at points x in a topo-model $M = (X, \mathcal{O}, v)$ in the following way:

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	and dually:	
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Sound and Complete Topo-Axiomatizations

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S4 is the topo-logic of all topological spaces (McKinsey & Tarski 1944).

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S4=Topo

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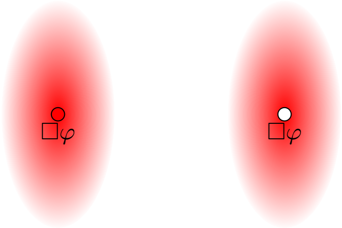
What about T_d-spaces (the learning spaces)?

T_d is not topo-definable.

Learnable spaces are not *topo*-definable.

Luckily, we can once again change the way we view $\Box$.



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M, x \models_d \neg\varphi & \text{iff} & \text{not } M, x \models_d \varphi \\
M, x \models_d \varphi \wedge \psi & \text{iff} & M, x \models_d \varphi \text{ and } M, x \models_d \psi \\
M, x \models_d \Box\varphi & \text{iff} & \exists U \in \tau (x \in U \;\&\; \forall y \in U - \{x\}\; M, y \models_d \varphi)
\end{array}
$$

and dually:

$$
M, x \models_d \Diamond\varphi \quad \text{iff} \quad \forall U \in \tau (x \in U \to \exists y \in U - \{x\}\; M, y \models_d \varphi)
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(K) $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$

K4=T_d

(4) $\Box\varphi \to \Box\Box\varphi$

K4 is the d-logic of all T_d-spaces.

KD45 Doxastic d-logic (Steinvold 2006)

Because independent reasons (e.g., Stalnaker) one may want B:=$\Box$ to be:

(K) $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$

(D) $\Box\varphi \rightarrow \neg\Box\neg\varphi$

(4) $\Box\varphi \rightarrow \Box\Box\varphi$

(5) $\neg\Box\varphi \rightarrow \Box\neg\Box\varphi$

Theorem (Steinsvold 2006)

KD45 is a sound and complete d-axiomatization of DSO spaces.

DSO stands for 'derived sets are open'. DSO are T_d-spaces (by 4), in which all derived sets are open (5), except that there are no open singletons (D).

QUESTIONS

But $DSO \subset T_d$.

So what do we talk about when we talk about beliefs in learning?

Should conjectures be interpreted as beliefs?

What if one restricts conjectures to only those which are ‘proper’ beliefs?

Outline

Introduction

Characterization of Learnability and Solvability

Constructive Order-driven Learning

Towards Epistemic Logic of Learnability

Intermediate (but Interesting) Conclusions

Conclusions

- Topological characterization of learnability & solvability in the limit.
- Universality of conditioning as a problem solving method.
- Use of stratification-like topological techniques.

Moreover:

- Learnable spaces are T_d.
- T_d-spaces are not topo-definable.
- Learnability is not topo-definable.
- Learnability cannot be expressed by solely topo-definable belief operators.
- The existing topo- and d-logics of belief are to fluffy to capture learnability.

THANK YOU!

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