Some Recent Results in Algebra & Logical Calculi Obtained Using Automated Reasoning

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[a]Results reported here are either the result of joint work, or the work of others associated with AR @ MCS @ ANL: Larry Wos, Bill McCune, Ken Kunen, Steve Winker, Bob Veroff, Ken Harris, Zac Ernst, John Slaney, Ted Ulrich, Bob Meyer, R. Padmanabhan *et al*.

Overview of Presentation

- Simple Equational Bases for Boolean Algebra
  - In the connectives + (disjunction) and n (negation)
  - In the connective | (Sheffer Stroke)
  - Pointers to recent work in other algebraic systems
- Simple Axiomatizations of Classical Propositional Logics
  - Classical Propositional Logic (in two sets of connectives)
  - Equivalential Fragment of Classical Propositional Logic
- Simple Axiomatizations of Non-Classical Propositional Logics
  - Strict Implicational fragments of modal logics S4, S5
  - Implicational fragments of relevance logics R, RM, E, Ł
- Time permitting: discussion of methods

Equational Bases for BA in + and n I

- In 1933, E.V. Huntington presented the following 3-basis for BA [10, 9]:

(Commutativity+)	$x + y = y + x$
(Associativity+)	$(x + y) + z = x + (y + z)$
(Huntington)	$n(n(x) + y) + n(n(x) + n(y)) = x$

- BA is usually presented in terms of $+, \cdot, n, 0, 1$. From Huntington's basis, 0, 1, and $\cdot$, with appropriate properties, can be established (*easy* for OTTER [20]).
- Shortly thereafter, Herbert Robbins asked whether the Huntington equation can be replaced with the following equation (which is shorter by one "n"):

(Robbins) $n(n(x + y) + n(x + n(y))) = x$

- The Robbins problem remained open for over 63 years, and attracted the attention of various people, including Tarski, and others [8], [2].

Equational Bases for BA in + and n II

- In 1979, Steve Winker, a student visiting Argonne, learned of the Robbins problem from Joel Berman. He and Larry Wos began to attack the problem.
- Larry Wos suggested looking for properties that force Robbins algebras to be Boolean. Winker [52] ingeniously found several such conditions (both "hand" and automated reasoning), including the following two relatively weak ones:
  1. $\exists c \exists d(c + d = c)$
  2. $\exists c \exists d(n(c + d) = n(c))$
- In 1996, Bill McCune [22] used an Argonne TP (EQP [21], a cousin of OTTER [20]) to prove that all Robbins algebras satisfy Winker's (2), above.
- This solved the long-standing Robbins problem. But, the machine proof of Winker's condition was not very easy for a human to follow or understand.
- Since McCune's discovery, several people (including myself [7]) have tried, in various ways, to make the EQP (and OTTER) proofs easier to digest [3].

Equational Bases for BA in + and n III

- It was thought that the (32-symbol) Robbins basis for BA was the simplest known, until I dug-up the following 23-symbol 2-basis for Boolean algebra reported (without proof) by Carew Meredith in 1968 [30, p. 228]:

(Meredith$_1$) $n(n(x) + y) + x = x$

(Meredith$_2$) $n(n(x) + y) + (z + y) = y + (z + x)$

- In 1966, Tarski [45] reported that BA does have *single* +, n axioms. Building on Tarski's work, Padmanabhan and Quackenbush [33] gave a method for constructing such axioms. But, their method yields ***long*** single axioms [23].
- Recently, Bill McCune [26] discovered a 22-symbol single axiom for BA:

(DN$_1$) $n(n(n(x + y) + z) + n(x + n(n(z) + n(z + u)))) = z$

- OPEN: do shorter single axioms (or bases!) for BA (in + and n) exist?

Equational Bases for BA in Sheffer's | I

- In 1913, Sheffer [42] gave the following 3-basis for Boolean algebra in terms of a single binary connective | (Sheffer's | is just NAND: $x \mid y = n(x) + n(y)$).

(Sheffer$_1$) $(x \mid x) \mid (x \mid x) = x$

(Sheffer$_2$) $x \mid (y \mid (y \mid y)) = x \mid x$

(Sheffer$_3$) $(x \mid (y \mid z)) \mid (x \mid (y \mid z)) = ((y \mid y) \mid x) \mid ((z \mid z) \mid x)$

- Meredith [28] (again, in obscurity, and rediscovered by me) simplified matters in 1969 by presenting the following (23-symbol) 2-basis for the same theory.

(Meredith$_3$) $(x \mid x) \mid (y \mid x) = x$

(Meredith$_4$) $x \mid (y \mid (x \mid z)) = ((z \mid y) \mid y) \mid x$

- Recently, Bob Veroff [50] established the following (17-symbol) 2-basis:

(Commutativity |) $x \mid y = y \mid x$

(Veroff$_{26a}$) $(x \mid y) \mid (x \mid (y \mid z)) = x$

Equational Bases for BA in Sheffer's | II

- The work of Tarski [45] and Padmanabhan & Quackenbush [33] also implies the existence of single axioms for BA in the Sheffer Stroke. But, as before, the |-single axioms generated by the methods of [33] are quite long [23].
- Recently, [53, 26] the following 15-symbol 1-bases were discovered.

(Sh$_1$) $(x \mid ((y \mid x) \mid x)) \mid (y \mid (z \mid x)) = y$

(Sh$_2$) $((y \mid (x \mid y)) \mid y) \mid (x \mid (z \mid y)) = x$

- It is also known [53, 26] that these |-single axioms are the *shortest possible*.
- Elegant axioms for groups [13, 19], lattices [25], loops [14, 15, 12], and other algebraic structures [24] have been discovered by the extended Argonne team.

Sheffer Stroke Single Axioms for Sentential Logic I

- In 1917, Nicod [32] showed[a] that the following 23-symbol formula (in Polish notation) is a single axiom for classical sentential logic (D is interpreted semantically as NAND, *i.e.*, the Sheffer stroke):

(N) $DDpDqrDDtDttDDsqDDpsDps$

- The only rule of inference for Nicod's single axiom system is the following, somewhat odd, detachment rule for D:

(D-Rule) From $DpDqr$ and p, infer r.

- Łukasiewicz [17, pp. 179–196] later showed that the following *substitution instance* (t/s) of Nicod's axiom (N) would suffice:

(Ł$_1$) $DDpDqrDDsDssDDsqDDpsDps$

[a] Actually, Nicod's original proofs are erroneous (as noted by Łukasiewicz in [17]). See Scharle's [41] for a rigorous proof of the completeness of Nicod's system.

Sheffer Stroke Single Axioms for Sentential Logic II

- Łukasiewicz's student Mordchaj Wajsberg [51, pp. 37–39] later discovered the following *organic*[a] 23-symbol single axiom for D:

(W) $DDpDqrDDDsrDDpsDpsDpDpq$

- Łukasiewicz later discovered another 23-symbol organic axiom:

($Ł_2$) $DDpDqrDDpDrpDDsqDDpsDps$

- Ken Harris and I have recently discovered many new 23-symbol single axioms, some of which are organic and have only 4 variables, *e.g.*,

(HF_1) $DDpDqrDDpDqrDDsrDDrsDps$

- We have also shown that *23 symbol axioms are the shortest possible*.

[a] A single axiom is *organic* if it contains no tautologous subformulae. (N) and (Ł) are *non*-organic, because they contain tautologous subformulae of the form $DxDxx$.

Single C-O Axioms for Classical Sentential Logic

- Meredith [27, 29] reports two 19-symbol single axioms for classical sentential logic (using only the rule of condensed detachment, or *modus ponens* for C) in terms of implication C and the constant O (semantically, O is "The False"):

$$CCCCCpqCrOstCCtpCrp$$
$$CCCpqCCOrsCCspCtCup$$

- Meredith [27, page 156] claims to have "almost completed a proof that no single axiom of (C,O) can contain less than 19 letters." As far as we know, no such proof was ever completed (that is, until now...).
- We have performed an exhaustive search/elimination of all (C,O) theorems with fewer than 19 symbols. We have proven Meredith's conjecture: *no single axiom of classical PL in (C,O) can contain less than 19 letters*.[a]

[a] The elimination of some (C,O) candidates relied on matrices generated using *stochastic local search* techniques (as described by Ted Ulrich in his [47, 49] and by Cipra [2]). Stochastic local search is very powerful in the context of implicational logics. It has led to *many* useful (small) models.

Single Axioms for The Equivalential Fragment of Classical Sentential Logic

- In 1933, Łukasiewicz [46, 250–277] showed (*lots* of hand calculations!) that (with MP for E as the sole rule) the shortest single axioms for the equivalential (E) fragment of classical propositional logic contain 11 symbols. He found 2 such axioms.
- In the 1950's, Meredith [29] discovered seven more 11-symbol single axioms for E.
- John Kalman [11], and his student J. Peterson [36, 37], did extensive work on the problem in the 1970's. They found one more 11-symbol single axiom, and they eliminated all but 7 of the remaining 640 11-symbol candidate single axioms.
- In 1977–1979, Wos, Winker, *et al* (all at Argonne) [55] worked on the remaining 7 candidates. They ruled-out all but three, and showed that two of these three were single axioms. This left the following (*and last*) remaining 11-symbol candidate:

(XCB) $EpEEEpqErqr$

- About a year ago, we (Wos, Dolph Ulrich, Fitelson [54]) proved that XCB *is* a single axiom for the equivalential calculus. The proof contains substitution instances with over 2000 symbols. This completes a 70-year study initiated by Łukasiewicz.

New Bases for C5

- In their classic paper [16], Lemmon, Meredith, Meredith, Prior, and Thomas present several axiomatizations (assuming only the rule of condensed detachment, or *modus ponens* for C) of the system C5, which is the strict-implicational fragment of the modal logic S5.
- Bases for C5 containing 4, 3, 2, and a single axiom are presented in [16]. The following 2-basis is the shortest of these bases. It contains 20 symbols, 5-variables, and 9 occurrences of the connective C.

$$Cpp$$
$$CCCCpqrqCCqsCtCps$$

- The following 21-symbol (6-variable, 10-C) single axiom (due to C.A. Meredith) for C5 is also reported in [16]:

$$CCCCCttpqCrsCCspCuCrp$$

New Bases for C5 (Cont'd)

- We (Ernst, Fitelson, Harris, Wos) searched both for new (hopefully, shorter than previously known) single axioms for C5 and for new 2-bases for C5.
- We discovered the following new 2-basis for C5, which is shorter than any previously known basis (indeed, it is as short as *any possible* basis — see below). It has 18 symbols, 4 variables, and 8 occurrences of C:

$$Cpp$$
$$CCpqCCCCqrsrCpr$$

- Moreover, we discovered the following new 21-symbol (6-variable, 10-C) single axiom for C5 (as well as 5 others, not given here):

$$CCCCpqrCCssqCCqtCuCpt$$

- *No formula with fewer than 21 symbols is a single axiom for C5*. And, *no basis for C5* ***whatsoever*** *has fewer than 18 symbols*. Results to appear in [5].

New Bases for C4

- C4 is the strict-implicational fragment of the modal logic S4 (and several other modal logics in the neighborhood of S4 — see Ulrich's [48]).
- As far as we know, the shortest known basis for C4 is due to Ulrich (see Ulrich's [48]), and is the following 25-symbol, 11-C, 3-axiom basis:

$$Cpp \qquad CCpqCrCpq \qquad CCpCqrCCpqCpr$$

- Anderson & Belnap [1, p. 89] state the finding of a (short) single axiom for C4 as an open problem (as far as we know, this has *remained* open). The following is a 21-symbol (6-variable, 10-C) single axiom for C4:

$$CCpCCqCrrCpsCCstCuCpt$$

- We have also the following 20-symbol 2-basis for C4:

$$CpCqq \qquad CCpCqrCCpqCsCpr$$

- *No formula with fewer than 21 symbols is a single axiom for C4*. And, *no basis for C4* ***whatsoever*** *has fewer than 20 symbols*. Results to appear in [5].

New Bases for $RM_{\rightarrow}$

- The "classical" relevance logic R-Mingle (RM) was first carefully studied by Dunn in the late 60's (*e.g.*, in [4]). Interestingly, the implicational fragment of R-Mingle ($RM_{\rightarrow}$) has an older history.
- $RM_{\rightarrow}$ was studied (albeit, unwittingly!) by Sobociński in the early 50's. Sobociński [43] discusses a two-designated-value-variant of Łukasiewicz's three-valued implication-negation logic (I'll call Sobociński's logic **S**). Sobociński leaves the axiomatization of $\mathbf{S}_{\rightarrow}$ as an open problem.
- Rose [39, 40] solved Sobociński's open problem, but his axiomatizations of $\mathbf{S}_{\rightarrow}$ are very complicated and highly redundant (see Parks' [34]).
- Meyer & Parks [31, 35] report: (*i*) an independent 4-basis for $\mathbf{S}_{\rightarrow}$, (*ii*) that $\mathbf{S}_{\rightarrow} = RM_{\rightarrow}$ (thus, a 4-basis for $RM_{\rightarrow}$); and (*iii*) that $RM_{\rightarrow}$ can be axiomatized by adding the following "unintelligible" 21-symbol formula to $R_{\rightarrow}$:

$$CCCCCpqqprCCCCCqppqrr$$

New Bases for $RM_{\rightarrow}$ (Cont'd)

- In other words, Meyer & Parks gave the following 5-basis for $RM_{\rightarrow}$:

$$Cpp \qquad CpCCpqq \qquad CCpqCCrpCrq \qquad CCpCpqCpq$$
$$CCCCCpqqprCCCCCqppqrr$$

- The reflexivity axiom Cpp is dependent in the above 5-basis. The remaining (independent) 4-basis is the Meyer-Parks basis for $RM_{\rightarrow}$.
- After much effort (and, with valuable assistance from Bob Veroff and Larry Wos), we (Ernst, Fitelson, Harris) discovered the following 13-symbol replacement for Parks' 21-symbol formula (& there are none shorter [6]):

$$CCCCCpqrCqprr$$

- The contraction axiom $CCpCpqCpq$ is dependent in our new 4-basis. The remaining (independent) *3*-basis for $RM_{\rightarrow}$ contains 31 symbols and 14 C's (the Meyer-Parks basis has 4 axioms, 48 symbols, and 22 C's):

$$CpCCpqq \qquad CCpqCCrpCrq \qquad CCCCCpqrCqprr$$

(Long) Single Axioms for Some Non-Classical Logics

- It was shown by Rezuş [38] (building on earlier seminal work of Tarksi and Łukasiewicz [18]) that the systems $E_{\rightarrow}$, $R_{\rightarrow}$, and $Ł_{\rightarrow}$ have single axioms. However, applying the methods of [38] yields very long, *in*organic single axioms. As far as we know, these axioms have never been explicitly written down. Here is a 69-symbol (17-variable!) single axiom for the implicational fragment of Łukasiewicz's infinite-valued logic $Ł_{\rightarrow}$ (obtained by Ken Harris, using the methods of [38]):

$Ł_{\rightarrow}$: $CCCfCgfCCCCCCCCCcdCCecCedCCaCbazzCCCCxyyCCyxxwwCCCCtuCutCutssCCqCrqpp$

- Single axioms of comparable length (*i.e.*, containing fewer than 75 symbols) can also be generated for the relevance logics $E_{\rightarrow}$ and $R_{\rightarrow}$ (omitted). Here's what we know about the shortest single axioms for the systems $E_{\rightarrow}$, $R_{\rightarrow}$, $Ł_{\rightarrow}$, and $RM_{\rightarrow}$:
  - The shortest single axiom for $E_{\rightarrow}$ has between 23 and 75 symbols.
  - The shortest single axiom for $R_{\rightarrow}$ has between 23 and 75 symbols.
  - The shortest single axiom for $Ł_{\rightarrow}$ has at most 69 symbols.
  - The shortest single axiom for $RM_{\rightarrow}$ (if there is one[a]) has at least 23 symbols.

[a] Methods of [18] and [38] do *not* apply to $RM_{\rightarrow}$, so whether $RM_{\rightarrow}$ has a single axiom remains open.

Our "Procedure" for Finding Elegant Axiomatizations

1. First, we wrote computer programs to generate a large list of candidate formulas which were to be tested as axioms. For most problems, it was practical to generate an exhaustive list of all formulas with up to twenty-one symbols (we can now do these through 23 symbols).
2. All the formulas in the list would be tested (using matrices) to see which are likely to be tautologies in the system in question.[a]
3. We immediately eliminated large numbers of formulas by applying known results about axiomatizations in the various systems. For example, as reported by Lemmon et. al., every axiomatization for C5 must contain a formula with Cpp as a (possibly improper) subformula [16]. Another useful

[a] We say 'likely to be tautologies' because C4 and C5 do not have finite characteristic matrices. Thus, we used matrices which validate all tautologies for the system, but also validate a small number of contingent formulas. This approach works surprisingly well in many systems.

result for this purpose is the Diamond-McKinsey theorem that no Boolean algebra can be axiomatized by formulas containing less than three distinct propositional letters [1, p. 83].

4. A set of formulas was selected from the list at random. Using either SEM [56] or a program written by the authors, we found a matrix that respects modus ponens, invalidates a known axiom-basis for the system, but validates the formulas selected from the list. Such a model suffices to show that the formulas are not single axioms for the system.
5. All the remaining formulas in the list were then tested against that matrix. Every formula validated by that matrix would be eliminated.
6. Steps 4 and 5 were repeated until the list of candidate formulas was down to a small number, or eliminated entirely.
7. We would then use Otter [20] (+strategies!) to attempt to prove a known axiom basis from each of the remaining candidates.

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