

2



Infinite Sets

2.1 Some Infinite Sets

Some sets have infinitely many members.

Think of the set of all the New Year's Eves from here to eternity.

Or if you don't believe in eternity, think of the set of all the spatial points between London and New York. (Since there will always be another point between any two distinct such points, there will be no end of them.)

Again, think of the set of all grammatical English sentences. (Since there is no word limit on the length of English sentences, we can always go on making longer sentences from shorter ones by such devices as adding 'John said that' to the beginning, or putting 'and then they had tea.' at the end.)

These are slightly messy examples. If you want a nice clean example of an infinite set, simply take the set of all the natural numbers, $\{0, 1, 2, 3, \dots\}$.

While we are on numbers, take care not to confuse numbers with the *numerals* that name them. (See Boxes 4 and 5.) Numerals are *words* like 'one' and 'two' or *symbols* like '1' and '2'. Numbers are the more abstract things that these numerals name. The English word 'two' is a

Box 4 Use and Mention

Philosophers are very fussy (because they often need to be) about distinguishing words from the things that they refer to. If you want to talk about the word rather than the thing, you must put the word in quotes to form a name of that word itself. Here are some examples that illustrate this device.

London contains ten million people, but 'London' contains six letters.

Jack is an unpopular person, but 'Jack' is a popular name.

Seven is an odd number, but 'seven' is an English word—a numeral.

{John, Paul, George, Ringo} is the same set as $\{x: x \text{ is a Beatle}\}$, but '{John, Paul, George, Ringo}' and ' $\{x: x \text{ is a Beatle}\}$ ' are two different names for that set.

On the left-hand side of these examples we *use* the names, on the right we *mention* them.

different word from the French word 'deux', but they both name the same number. Again, the Arabic '2' is a different symbol from the Roman 'II' but they also both name the same number. Numerals are signs used in specific representation systems. Numbers themselves are timeless entities that transcend the perspective of any given system of representation. (See Box 6.)

2.2 Different Kinds of Numbers

The most basic numbers are the *natural numbers*: 0, 1, 2, 3, ...

If we add the negative whole numbers to the natural numbers, then we get the *integers*: ... -3, -2, -1, 0, 1, 2, 3 ...

In addition to the integers, we also want to recognize various kinds of intermediate numbers, numbers that fall between the integers.

The simplest are the *rational numbers*, namely those that can be expressed as fractions of the form p/q , where p and q are integers.

But we also need to recognize further numbers that are not rational.

For example, $\sqrt{2}$ is not rational. There is no way to express $\sqrt{2}$ in the form p/q where p and q are integers. (See Box 7.)

Similarly, π (the ratio of a circle's circumference to its diameter) is not rational. It cannot be expressed as p/q with integral p and q either.

Many other numbers are similarly irrational.

The *real numbers* comprise both the rational and irrational numbers.

Any real number can be represented by an infinitely long decimal expansion: e.g. 23.17564839...

Box 5 Types and Tokens

cat cat

Question. How many words were there in the previous line? *Answer.* One word *type*, but two *tokens* of that type.

The term “‘cat’ ” can refer either to the type word or to some specific token of it.

Thus: ‘cat’ occurs often in children’s stories. Here I use “‘cat’ ” to refer to a word type.

But now consider: the first ‘cat’ at the beginning of this Box could have been written with a capital letter. Here I use “‘cat’ ” to refer to a specific token of the relevant type.

(Note how I have to use double quotes—“‘cat’ ”—to mention the name of the original word, that is, the name that we formed by putting that original word in single quotes.)

Box 6 The Reality of Numbers

As with sets, it is possible to doubt whether numbers really exist. If they are outside space and time, and have no causal impact on anything, do we really need to believe in them? Some philosophers are indeed inclined to dismiss numbers, along with sets, as no more than useful fictions. But, as before, we can bypass this issue here, and think of ourselves as exploring what properties numbers *would* have, *if* they existed. Even those who are suspicious of numbers will do well to understand their workings, so to speak.

In this format, we can distinguish the rational numbers from the irrational ones by the fact that the rational numbers will eventually display some recurring sequence of digits. So for example, $1/11$ is $0.090909\dots$ and $2/7$ is $0.285714285714285714\dots$ (See the Exercises for some hints about how to show that the rational numbers are just those whose decimal expansions recur.)

2.3 Two Senses of ‘More’

Here is a good question. Are there more natural numbers than even numbers?

In one obvious sense the answer must be yes. The set of even numbers $\{0, 2, 4, 6, \dots\}$ is a *proper subset* of the set of natural numbers $\{0, 1, 2, 3, \dots\}$. The latter set contains all the members of the former set and then some. There are plenty of natural numbers that aren’t even, but no even numbers that aren’t natural.

Box 7 $\sqrt{2}$ is Irrational

Suppose (for the sake of a 'reductio ad absurdum' proof) that $\sqrt{2}$ is rational and so can be represented as p/q , where p and q are integers, and suppose further that p and q have no common factors, that is, that all cancelling has been done. Then it follows:

$$\sqrt{2} = p/q$$

$$2 = p^2/q^2$$

$$2q^2 = p^2$$

So p must be an even number (since its square is an even number). So, for some integer r , p must be $2r$. So

$$p^2 = 4r^2$$

And, since we already know that $2q^2 = p^2$, it follows that

$$q^2 = 2r^2$$

So q must be an even number too. But now q and p are both even, which contradicts the supposition that $\sqrt{2}$ is rational and represented as p/q with no common factors. So by reductio we can conclude that $\sqrt{2}$ is irrational.

When the Greeks first discovered that $\sqrt{2}$ is irrational, it freaked them out. They knew from Pythagoras' theorem that $\sqrt{2}$ is the length of the hypotenuse of a right-angled triangle whose other sides are each of length 1. But the irrationality of $\sqrt{2}$ means that there can be no unit of length that will fit exactly q times into these short sides and p times into the hypotenuse (for if there were, then $\sqrt{2}$ would equal p/q). To the Greeks, this seemed to contradict the very idea of length. It is said that the Greek mathematicians who first proved the irrationality of $\sqrt{2}$ tried to keep their discovery a secret.

But in a different sense the answer is no. The even numbers can be *paired up one-to-one* with the naturals. In this sense there are just as many even numbers as natural numbers.

0	2	4	6	8	...
0	1	2	3	4	...

This mapping gives a unique even number for every natural number, and vice versa.

There is no contradiction here. We can distinguish two senses in which set A can contain ‘more members’ than set B. In the first sense (call it the ‘subset’ sense), it simply means that B is a proper subset of A. In the second sense (the ‘pairing’ sense), it means rather that any attempt to pair the members of A one-to-one with those of B will leave some members of A unpaired.

There are more natural numbers than even numbers in the subset sense, but not in the pairing sense—for the pairing illustrated above succeeds in matching every natural number with its own even number.

When we are dealing with finite sets, the two senses of ‘more’ coincide. If a finite set B is a proper subset of finite set A, then the As can’t all be paired up one-to-one with the Bs, for there won’t be enough Bs—any attempted pairing will leave some extra As unpaired.

But with infinite sets, B can be a proper subset of A, and still be paired up one-to-one with the As—for now the Bs won’t automatically run out before we get to the end of the As.

This is in fact a defining characteristic of infinite sets. The members of any infinite set, but of no finite set, can be paired up one-to-one with the members of some of its proper subsets.

2.4 Denumerability

The odd numbers can also be paired one-to-one with the natural numbers.

1	3	5	7	9	...
0	1	2	3	4	...

So can the squared whole numbers.

0	1	4	9	16	...
0	1	2	3	4	...

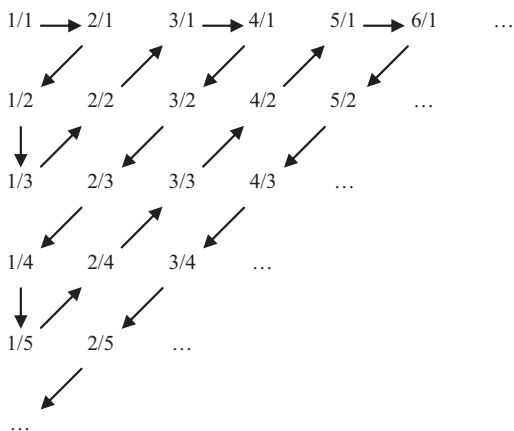
And all the integers.

0	-1	+1	-2	+2	...
0	1	2	3	4	...

What about the rational numbers? At first sight it might seem that there are too many. There really are an awful lot. In particular, given any two rational numbers, however close together, there will always be another rational number in between them. (Mathematicians call this property ‘density’.) You might think that this would block any attempt to line them up with the natural numbers.

Surprisingly, however, the rational numbers can also be paired up one-to-one with the natural numbers. To see this, consider the following grid. It clearly contains all the rational numbers. And the arrows indicate a systematic way of going through the grid in sequence and thereby placing the rational numbers in a numerical order.¹

¹ A little complication. If we list the rational numbers as in the diagram below, any given rational number will recur in different guises at different points in the list. For example, we will not only have $1/2$, but later on $2/4$, $3/6$, and so on. Since these are all the same rational number, just written in different ways, our list won’t really pair each rational number with a *unique* natural number. The remedy is to complicate the listing procedure a bit—before writing down the n^{th} rational number, check that it hasn’t already occurred in the list, and throw it away if it has.



Whenever the members of a set can be paired one-to-one with the natural numbers, we say the set is *denumerable*. A denumerable set is one that can be placed in a numerical list. A numerical list, if you think about it, just is a pairing of the listed items with the natural numbers—the first in the list with 1, the second with 2, and so on.

2.5 More Denumerable Sets

Many unruly-looking sets can be shown to be denumerable.

Take the set of all rectangles with rational length and breadth, for example. Each of these is defined by two rational numbers. Given that we can place all the rational numbers themselves in a numerical list, by the grid trick above, we can thus equate each of these pairs of rational numbers with a pair of *natural* numbers. And then we can apply the grid technique once more, to place these pairs of natural numbers themselves in a numerical list. This will then amount to a numerical list of the rectangles we started with.

Or take the set of all English sentences. To place these in a numerical list, consider all finitely long strings of English letters (counting a space as a 27th letter). Now order the one-letter strings alphabetically, then the two-letter strings, and so on. Now go through the resulting list and throw away all the strings which don't make sense as English sentences. You'll be left with a numerical list of English sentences.

There are many similar examples of denumerable sets.

2.6 The Non-Denumerability of the Real Numbers

We have just seen that many complicated-looking infinite sets turn out to be denumerable. Does this hold for all infinite sets? Our surprising success at pairing the rational numbers and other unpromising-looking sets with the natural numbers might make you think that a similar trick can be pulled with all infinite sets. But that would be a mistake. The *real numbers* cannot be paired one-to-one with the natural numbers. They are *non*-denumerable. Indeed the reals between 0 and 1, or in any finite interval, are non-denumerable.

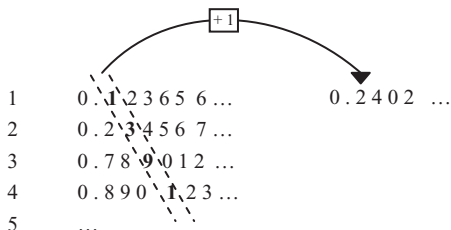
To show this, suppose (for the sake of another *reductio* argument) that the reals between 0 and 1 *were* denumerable. Then they could be paired up with the natural numbers in some way. To illustrate, suppose the pairing starts as in the list below. (This is just for illustration—the argument will work whatever the pairing.)

- | | |
|---|-------------|
| 1 | 0.123456... |
| 2 | 0.234567... |
| 3 | 0.987654... |
| 4 | 0.976543... |

Now construct a new number according to the following rule: make the first digit one more than the first digit of the first number in this

Diagram illustrating the addition of 1 to a number in base 10. A box labeled $+1$ is shown above a sequence of numbers. A curved arrow points from the box to the first number, $0.123656\dots$, which has a dashed line through it. Another curved arrow points from the box to the second number, $0.234567\dots$, which has a solid arrow pointing to it. The sequence of numbers is:

- 1 $0.123656\dots$
- 2 $0.234567\dots$
- 3 $0.789012\dots$
- 4 $0.890123\dots$
- 5 \dots



This is Cantor's famous diagonal argument. It shows that there are more real numbers than natural numbers *even in the one-to-one pairing*

² There is a little complication in this diagonal proof too. Some real numbers have two decimal representations. Consider for example $0.999\dots = 3 \times 0.333\dots = 3 \times 1/3 = 1$. This shows that $0.999\dots$ and $1\dots$ are the same real number written in different ways. And this might make you worry that Cantor's argument only proves that there is a 'diagonal representation' that isn't in the original list of decimal representations, not that there is a real number that isn't among the numbers named by that list—for maybe the 'diagonal representation' is just an alternative name for one of the numbers already listed.

Well, it would be interesting enough to know that the set of decimal representations is itself non-denumerable, even if the real numbers themselves aren't. But in any case it is easy enough to tighten the proof so as to plug this hole. One of the Exercises at the end of Chapter 3 covers this.

sense of ‘more’. If you try to pair up the reals with the naturals you will always have some real number left over. Given any supposed listing of the reals, it is always possible to construct another real number that isn’t in that list.

2.7 The Abundance of the Real Numbers

The reals are very abundant indeed. To get some feel for this, recall that the real numbers are represented by *infinitely* long decimal strings, including strings that display no recurring patterns. The other entities we have been dealing with (rational numbers, sentences, ...) can all be represented in finite terms. This doesn’t stop there being infinitely many rational numbers or sentences—finite representations can get longer and longer. But once we switch to *infinitely* long strings of digits, we are dealing with a quite different order of plurality.

The example of the reals shows that infinite sets come in different sizes. There is the size shared by all the denumerable sets. But the real numbers are bigger again. In the next chapter we shall explore the way in which different infinite sets can have different sizes in this way.

FURTHER READING

Numbers: A Very Short Introduction by Peter Higgins (Oxford University Press 2011) explains the different kinds of numbers.

The last two chapters of Eric Steinhardt's *More Precisely: The Math You Need To Do Philosophy* (Broadview Press 2009) deal with infinite sets and the variety of infinite numbers.

An Introduction to the Philosophy of Mathematics by Mark Colyvan (Cambridge University Press 2012) is a short and punchy introduction to the philosophical issues raised by numbers and mathematical objects.

James Robert Brown's *Philosophy of Mathematics: An Introduction to a World of Proofs and Pictures* (Routledge 1999) is another lively introduction to this area.

EXERCISES

1. Write a sentence that both uses and mentions the word 'philosophy'. Write a sentence that both uses and mentions some other word. Say where in the two sentences the relevant words are used and where mentioned.
2. 77
How many token numerals are on the previous line? How many type numerals?
How many natural numbers are less than 10? How many Arabic type numerals are written with one digit?
3. Show how all the integral multiples of 5 (positive and negative) can be paired one-to-one with the natural numbers.
4. Which of the following are subsets of the natural numbers?
 - (a) the squares of the natural numbers
 - (b) the square roots of the natural numbers

- (c) the positive whole numbers less than 10 million
- (d) the rational numbers

- 5*. Show that any rational number p/q , with p and q integers, will have a decimal expansion that eventually recurs. (Hint: think about what will happen as you generate the decimal expansion by dividing q into p .)
- 6*. Show that any decimal number that terminates with a recurring part is equal to some rational number. (Hint: first separate the recurring part, then multiply it by 10^k , where k is the number of digits in the recurring part, then see what happens when you subtract the original recurring part from this number.)

(*Exercises with starred numerals are more difficult.)