2 TRUTH FUNCTIONAL CONNECTIVES

1. INTRODUCTION

As noted earlier, an argument is valid or invalid purely in virtue of its form. The form of an argument is a function of the arrangement of the terms in the argument, where the logical terms play a primary role. However, as noted earlier, what counts as a logical term, as opposed to a descriptive term, is not absolute. Rather, it depends upon the level of logical analysis we are pursuing.

In the previous chapter we briefly examined one level of logical analysis, the level of syllogistic logic. In syllogistic logic, the logical terms include 'all', 'some', 'no', 'are', and 'not', and the descriptive terms are all expressions that denote classes.

In the next few chapters, we examine a different branch of logic, which represents a different level of logical analysis; specifically, we examine sentential logic (also called propositional logic and statement logic). In sentential logic, the logical terms are truth-functional statement connectives, and nothing else.

2. STATEMENT CONNECTIVES

We begin by defining statement connective, or what we will simply call a connective.

> A **(statement)** ***connective*** is an expression with one or more blanks (places) such that, whenever the blanks are filled by statements the resulting expression is also a statement.

In other words, a (statement) connective takes one or more smaller statements and forms a larger statement. The following is a simple example of a connective.

___________ and ____________

To say that this expression is a connective is to say that if we fill each blank with a statement then we obtain another statement. The following are examples of statements obtained in this manner.

(e1) snow is white **and** grass is green
(e2) all cats are felines **and** some felines are not cats
(e3) it is raining **and** it is sleeting

Notice that the blanks are filled with statements and the resulting expressions are also statements.

The following are further examples of connectives, which are followed by particular instances.

(c1) it is not true that ____________________
(c2) the president believes that ____________
(c3) it is necessarily true that _____________

(c4) ___________ or ___________
(c5) if ___________ then ___________
(c6) ___________ only if ___________
(c7) ___________ unless ___________

(c8) ___________ if ___________; otherwise ___________
(c9) ___________ unless ___________ in which case ___________

(i1) it is not true that all felines are cats
(i2) the president believes that snow is white
(i3) it is necessarily true that 2+2=4

(i4) it is raining or it is sleeting
(i5) if it is raining then it is cloudy
(i6) I will pass only if I study

(i7) I will play tennis unless it rains
(i8) I will play tennis if it is warm; otherwise I will play racquetball
(i9) I will play tennis unless it rains in which case I will play squash

Notice that the above examples are divided into three groups, according to how many blanks (places) are involved. This grouping corresponds to the following series of definitions.

> A ***one-place*** **connective** is a connective with one blank.

> A ***two-place*** **connective** is a connective with two blanks.

> A ***three-place*** **connective** is a connective with three blanks.

etc.

At this point, it is useful to introduce a further pair of definitions.

> A ***compound statement*** is a statement that is constructed from one or more smaller statements by the application of a statement connective.

> A ***simple statement*** is a statement that is *not* constructed out of smaller statements by the application of a statement connective.

We have already seen many examples of compound statements. The following are examples of *simple statements*.

(s1) snow is white
(s2) grass is green
(s3) I am hungry
(s4) it is raining
(s5) all cats are felines
(s6) some cats are pets

Note that, from the viewpoint of sentential logic, all statements in syllogistic logic are simple statements, which is to say that they are regarded by sentential logic as having no internal structure.

In all the examples we have considered so far, the constituent statements are all simple statements. A connective can also be applied to compound statements, as illustrated in the following example.

it is not true that all swans are white,
and
the president believes that all swans are white

In this example, the two-place connective '...and...' connects the following two statements,

it is not true that all swans are white

the president believes that all swans are white

which are themselves compound statements. Thus, in this example, there are three connectives involved:

it is not true that...

...and...

the president believes that...

The above statement can in turn be used to form an even larger compound statement. For example, we combine it with the following (simple) statement, using the two-place connective 'if...then...'.

the president is fallible

We accordingly obtain the following compound statement.

IF it is not true that all swans are white,
AND the president believes that all swans are white,
THEN the president is fallible

There is no *theoretical* limit on the complexity of compound statements constructed using statement connectives; *in principle*, we can form compound statements that are as long as we please (say a billion miles long!). However, there are *practical* limits to the complexity of compound statements, due to the limitation of space and time, and the limitation of human minds to comprehend excessively long and complex statements. For example, I doubt very seriously whether any human can understand a statement that is a billion miles long (or even one mile long!) However, this is a practical limit, not a theoretical limit.

By way of concluding this section, we introduce terminology that is often used in sentential logic. Simple statements are often referred to as *atomic statements*, or simply atoms, and by analogy, compound statements are often referred to as *molecular statements*, or simply molecules.

The analogy, obviously, is with chemistry. Whereas *chemical* atoms (hydrogen, oxygen, etc.) are the smallest *chemical* units, *sentential* atoms are the smallest *sentential* units. The analogy continues. Although the word 'atom' literally means "that which is *indivisible*" or "that which has *no parts*", we know that the chemical atoms do have parts (neutrons, protons, etc.); however, these parts are not chemical in nature. Similarly, atomic sentences have parts, but these parts are not sentential in nature. These further (sub-atomic) parts are the topic of later chapters, on predicate logic.

3. TRUTH-FUNCTIONAL STATEMENT CONNECTIVES

In the previous section, we examined the *general* class of (statement) connectives. At the level we wish to pursue, sentential logic is not concerned with *all* connectives, but only special ones – namely, the *truth-functional* connectives.

Recall that a statement is a sentence that, when uttered, is either true or false. In logic it is customary to refer to *truth* and *falsity* as *truth values*, which are respectively abbreviated T and F. Furthermore, if a statement is true, then we say its *truth value* is T, and if a statement is false, then we say that its *truth value* is F. This is summarized as follows.

The **truth value** of a true statement is **T**.

The **truth value** of a false statement is **F**.

The truth value of a statement (say, 'it is raining') is analogous to the weight of a person. Just as we can say that the weight of John is 150 pounds, we can say that the truth value of 'it is raining' is T. Also, John's weight can *vary* from day to day; one day it might be 150 pounds; another day it might be 152 pounds. Similarly, for some statements at least, such as 'it is raining', the truth value can vary from occasion to occasion. On one occasion, the truth value of 'it is raining' might be T; on another occasion, it might be F. The difference between weight and truth-value is quantitative: whereas weight can take infinitely many values (the positive real numbers), truth value can only take two values, T and F.

The analogy continues. Just as we can apply *functions* to numbers (addition, subtraction, exponentiation, etc.), we can apply functions to truth values. Whereas the former are *numerical* functions, the latter are *truth-functions*.

In the case of a numerical function, like addition, the input are numbers, and so is the output. For example, if we input the numbers 2 and 3, then the output is 5. If we want to learn the addition function, we have to learn what the output number is for any two input numbers. Usually we learn a tiny fragment of this in elementary school when we learn the addition tables. The addition tables tabulate the output of the addition function for a few select inputs, and we learn it primarily by rote.

Truth-functions do not take numbers as input, nor do they produce numbers as output. Rather, truth-functions take truth values as input, and they produce truth values as output. Since there are only two truth values (compared with infinitely many numbers), learning a truth-function is considerably simpler than learning a numerical function.

Just as there are two ways to learn, and to remember, the addition tables, there are two ways to learn truth-function tables. On the one hand, you can simply memorize it (two plus two is four, two plus three is five, etc.) On the other hand, you can master the underlying concept (what are you doing when you add two numbers together?) The best way is probably a combination of these two techniques.

We will discuss several examples of truth functions in the following sections. For the moment, let's look at the definition of a truth-functional connective.

> A statement connective is ***truth-functional*** if and only if the truth value of *any* compound statement obtained by applying that connective is a function of (is completely determined by) the individual truth values of the constituent statements that form the compound.

This definition will be easier to comprehend after a few examples have been discussed. The basic idea is this: suppose we have a statement connective, call it +, and suppose we have any two statements, call them S_1 and S_2. Then we can form a compound, which is denoted S_1+S_2. Now, to say that the connective + is truth-functional is to say this: if we know the truth values of S_1 and S_2 individually, then we *automatically know*, or at least we can *compute*, the truth value of S_1+S_2. On the other hand, to say that the connective + is *not* truth-functional is to say this: merely knowing the truth values of S_1 and S_2 does not automatically tell us the truth value of S_1+S_2. An example of a connective that is *not* truth-functional is discussed later.

4. CONJUNCTION

The first truth-functional connective we discuss is conjunction, which corresponds to the English expression ‘and’.

[Note: In traditional grammar, the word ‘conjunction’ is used to refer to *any* two-place statement connective. However, in logic, the word ‘conjunction’ refers exclusively to one connective – ‘and’.]

Conjunction is a two-place connective. In other words, if we have two statements (simple or compound), we can form a compound statement by combining them with ‘and’. Thus, for example, we can combine the following two statements

it is raining
it is sleeting

to form the compound statement

it is raining **and** it is sleeting.

In order to aid our analysis of logical form in sentential logic, we employ various symbolic devices. First, we abbreviate *simple* statements by upper case Roman letters. The letter we choose will usually be suggestive of the statement that is abbreviated; for example, we might use ‘R’ to abbreviate ‘it is raining’, and ‘S’ to abbreviate ‘it is sleeting’.

Second, we use special symbols to abbreviate (truth-functional) connectives. For example, we abbreviate conjunction (‘and’) by the ampersand sign (‘&’). Putting these abbreviations together, we abbreviate the above compound as follows.

R & S

Finally, we use parentheses to punctuate compound statements, in a manner similar to arithmetic. We discuss this later.

A word about terminology, R&S is called a *conjunction*. More specifically, R&S is called *the conjunction of* R and S, which individually are called *conjuncts*. By analogy, in arithmetic, x+y is called the *sum of* x and y, and x and y are individually called *summands*.

Conjunction is a truth-functional connective. This means that if we know the truth value of each conjunct, we can simply compute the truth value of the conjunction. Consider the simple statements R and S. Individually, these can be true or false, so in combination, there are four cases, given in the following table.

	R	S
case 1	T	T
case 2	T	F
case 3	F	T
case 4	F	F

In the first case, both statements are true; in the fourth case, both statements are false; in the second and third cases, one is true, the other is false.

Now consider the conjunction formed out of these two statements: R&S. What is the truth value of R&S in each of the above cases? Well, it seems plausible that the conjunction R&S is true *if* both the conjuncts are true individually, and R&S is false if either conjunct is false. This is summarized in the following table.

	R	S	R&S
case 1	T	T	T
case 2	T	F	F
case 3	F	T	F
case 4	F	F	F

The information contained in this table readily generalizes. We do not have to regard 'R' and 'S' as standing for *specific* statements. They can stand for any statements whatsoever, and this table still holds. No matter what R and S are specifically, if they are both true (case 1), then the conjunction R&S is also true, but if one or both are false (cases 2-4), then the conjunction R&S is false.

We can summarize this information in a number of ways. For example, each of the following statements summarizes the table in more or less ordinary English. Here, $\mathcal{A}$ and $\mathcal{B}$ stand for arbitrary statements.

A conjunction $\mathcal{A}\&\mathcal{B}$ is true
if and only if
both conjuncts are true.

A conjunction $\mathcal{A}\&\mathcal{B}$ is true if both conjuncts are true;
otherwise, it is false.

We can also display the truth function for conjunction in a number of ways. The following three tables present the truth function for conjunction; they are followed by three corresponding tables for multiplication.

$\mathcal{A}$	$\mathcal{B}$	$\mathcal{A}\&\mathcal{B}$
T	T	T
T	F	F
F	T	F
F	F	F

$\mathcal{A}$	&	$\mathcal{B}$
T	T	T
T	F	F
F	F	T
F	F	F

&	T	F
T	T	F
F	F	F

a	b	a×b
1	1	1
1	0	0
0	1	0
0	0	0

a	×	b
1	1	1
1	0	0
0	0	1
0	0	0

×	1	0
1	1	0
0	0	0

Note: The middle table is obtained from the first table simply by superimposing the three columns of the first table. Thus, in the middle table, the truth values of $\mathcal{A}$ are all under the $\mathcal{A}$, the truth values of $\mathcal{B}$ are under the $\mathcal{B}$, and the truth values of $\mathcal{A}\&\mathcal{B}$ are the &. Notice, also, that the final (output) column is also shaded, to help distinguish it from the input columns. This method saves much space, which is important later.

We can also express the content of these tables in a series of statements, just like we did in elementary school. The conjunction truth function may be conveyed by the following series of statements. Compare them with the corresponding statements concerning multiplication.

(1)	T & T = T	$1 \times 1 = 1$
(2)	T & F = F	$1 \times 0 = 0$
(3)	F & T = F	$0 \times 1 = 0$
(4)	F & F = F	$0 \times 0 = 0$

For example, the first statement may be read "T ampersand T is T" (analogously, "one times one is one"). These phrases may simply be memorized, but it is better to understand what they are about – namely, conjunctions.

5. DISJUNCTION

The second truth-functional connective we consider is called *disjunction*, which corresponds roughly to the English 'or'. Like conjunction, disjunction is a two-place connective: given any two statements S_1 and S_2, we can form the compound statement 'S_1 or S_2'. For example, beginning with the following simple statements,

(s1)	it is raining	R
(s2)	it is sleeting	S

we can form the following compound statement.

(c)	it is raining **or** it is sleeting	$R \vee S$

The symbol for disjunction is '$\vee$' (wedge). Just as R&S is called the *conjunction* of R and S, $R \vee S$ is called the *disjunction* of R and S. Similarly, just as the constituents of a conjunction are called *conjuncts*, the constituents of a disjunction are called *disjuncts*.

In English, the word 'or' has at least two different meanings, or senses, which are respectively called the **exclusive sense** and the **inclusive sense**. The exclusive sense is typified by the following sentences.

(e1) would you like a baked potato, OR French fries
(e2) would you like squash, OR beans

In answering these questions, you cannot choose both disjuncts; choosing one disjunct *excludes* choosing the other disjunct.

On the other hand, the inclusive sense of disjunction is typified by the following sentences.

(i1) would you like coffee or dessert
(i2) would you like cream or sugar with your coffee

In answering these questions, you can choose both disjuncts; choosing one disjunct does not exclude choosing the other disjunct as well.

Latin has two different disjunctive words, ‘vel’ (inclusive) and ‘aut’ (exclusive). By contrast, English simply has one word ‘or’, which does double duty. This problem has led the legal profession to invent the expression ‘and/or’ to use when inclusive disjunction is intended. By using ‘and/or’ they are able to avoid ambiguity in legal contracts.

In logic, the *inclusive sense* of ‘or’ (the sense of ‘vel’ or ‘and/or’) is taken as basic; it is symbolized by wedge ‘$\vee$’ (suggestive of ‘v’, the initial letter of ‘vel’). The truth table for $\vee$ is given as follows.

$\mathcal{A}$	$\mathcal{B}$	$\mathcal{A}\vee\mathcal{B}$
T	T	T
T	F	T
F	T	T
F	F	F

$\mathcal{A}$	$\vee$	$\mathcal{B}$
T	T	T
T	T	F
F	T	T
F	F	F

$\vee$	T	F
T	T	T
F	T	F

The information conveyed in these tables can be conveyed in either of the following statements.

A disjunction $\mathcal{A}\vee\mathcal{B}$ is false
if and only if
both disjuncts are false.

A disjunction $\mathcal{A}\vee\mathcal{B}$ is false if both disjuncts are false;
otherwise, it is true.

The following is an immediate consequence, which is worth remembering.

If $\mathcal{A}$ is true, then so is $\mathcal{A}\vee\mathcal{B}$,
regardless of the truth value of $\mathcal{B}$.

If $\mathcal{B}$ is true, then so is $\mathcal{A}\vee\mathcal{B}$,
regardless of the truth value of $\mathcal{A}$.

6. A STATEMENT CONNECTIVE THAT IS NOT TRUTH-FUNCTIONAL

Conjunction (&) and disjunction ($\vee$) are both truth-functional connectives. In the present section, we discuss a connective that is not truth-functional – namely, the connective 'because'.

Like conjunction ('and') and disjunction ('or'), 'because' is a two-place connective; given any two statements S_1 and S_2, we can form the compound statement 'S_1 because S_2'. For example, given the following simple statements

(s1)	I am sad	S
(s2)	it is raining	R

we can form the following compound statements.

(c1)	I am sad **because** it is raining	S because R
(c2)	it is raining **because** I am sad	R because S

The simple statements (s1) and (s2) can be individually true or false, so there are four possible combinations of truth values. The question is, for each combination of truth values, what is the truth value of each resulting compound.

First of all, it seems fairly clear that if either of the simple statements is false, then the compound is false. On the other hand, if both statements are true, then it is not clear what the truth value of the compound is. This is summarized in the following *partial* truth table.

S	R	S because R	R because S
T	T	?	?
T	F	F	F
F	T	F	F
F	F	F	F

In the above table, the question mark (?) indicates that the truth value is unclear.

Suppose both S ('I am sad') and R ('it is raining') are true. What can we say about the truth value of 'S because R' and 'R because S'? Well, at least in the case of

it is raining because I am sad,

we can safely assume that it is false (unless the speaker in question is God, in which case all bets are off).

On the other hand, in the case of

I am sad because it is raining,

we cannot say whether it is true, or whether it is false. Merely knowing that the speaker is sad and that it is raining, we do not know whether the rain is *responsible* for the sadness. It might be, it might not. Merely knowing the individual truth values of S ('I am sad') and R ('it is raining'), we do not automatically know the truth value of the compound ‘I am sad because it is raining’; additional information (of a complicated sort) is needed to decide whether the compound is true or false. In other words, ‘because’ is not a truth-functional connective.

Another way to see that ‘because’ is *not* truth-functional is to suppose to the contrary that it *is* truth-functional. If it is truth-functional, then we can replace the question mark in the above table. We have only two choices. If we replace ‘?’ by ‘T’, then the truth table for ‘R because S’ is identical to the truth table for R&S. This would mean that for any statements $\mathcal{A}$ and $\mathcal{B}$, ‘$\mathcal{A}$ because $\mathcal{B}$’ says no more than ‘$\mathcal{A}$ and $\mathcal{B}$’. This is absurd, for that would mean that both of the following statements are true.

grass is green *because* 2+2=4
2+2=4 *because* grass is green

Our other choice is to replace ‘?’ by ‘F’. This means that the output column consists entirely of F's, which means that ‘$\mathcal{A}$ because $\mathcal{B}$’ is *always false*. This is also absurd, or at least implausible. For surely some statements of the form ‘$\mathcal{A}$ because $\mathcal{B}$’ are true. The following might be considered an example.

grass is green *because* grass contains chlorophyll

7. NEGATION

So far, we have examined three two-place connectives. In the present section, we examine a one-place connective, negation, which corresponds to the word ‘not’.

If we wish to deny a statement, for example,

it is raining,

the easiest way is to insert the word ‘not’ in a strategic location, thus yielding

it is **not** raining.

We can also deny the original statement by prefixing the whole sentence by the modifier

it is not true that

to obtain

it is not true that it is raining

The advantage of the first strategy is that it produces a colloquial sentence. The advantage of the second strategy is that it is simple to apply; one simply prefixes the statement in question by the modifier, and one obtains the denial. Furthermore, the second strategy employs a statement connective. In particular, the expression

it is not true that ______________

meets our criterion to be a one-place connective; its single blank can be filled by any statement, and the result is also a statement.

This one-place connective is called *negation*, and is symbolized by '∼' (tilde), which is a stylized form of 'n', short for negation. The following are variant negation expressions.

it is false that __________________
it is not the case that ____________

Next, we note that the negation connective (∼) is truth-functional. In other words, if we know the truth value of a statement S, then we automatically know the truth value of the negation ∼S; the truth value of ∼S is simply the opposite of the truth value of S.

This is plausible. For ∼S denies what S asserts; so if S is in fact false, then its denial (negation) is true, and if S is in fact true, then its denial is false. This is summarized in the following truth tables.

$\mathcal{A}$	$\sim\mathcal{A}$
T	F
F	T

$\sim$	$\mathcal{A}$
F	T
T	F

In the second table, the truth values of $\mathcal{A}$ are placed below the $\mathcal{A}$, and the resulting truth values for $\sim\mathcal{A}$ are placed below the tilde sign (∼). The right table is simply a compact version of the left table. Both tables can be summarized in the following statement.

$\boxed{\sim\mathcal{A} \text{ has the opposite truth value of } \mathcal{A}.}$

8. THE CONDITIONAL

In the present section, we introduce one of the two remaining truth-functional connectives that are customarily studied in sentential logic – the *conditional connective*, which corresponds to the expression

if ___________, then ___________.

The conditional connective is a two-place connective, which is to say that we can replace the two blanks in the above expression by any two statements, then the resulting expression is also a statement.

For example, we can take the following simple statements.

(1) I am relaxed
(2) I am happy

and we can form the following conditional statements, using if-then.

(c1) **if** I am relaxed, **then** I am happy
(c2) **if** I am happy, **then** I am relaxed

The symbol used to abbreviate if-then is the arrow (→), so the above compounds can be symbolized as follows.

(s1) $R \rightarrow H$
(s2) $H \rightarrow R$

Every conditional statement divides into two constituents, which do *not* play equivalent roles (in contrast to conjunction and disjunction). The constituents of a conditional $\mathcal{A} \rightarrow \mathcal{C}$ are respectively called the *antecedent* and the *consequent*. The word 'antecedent' means "that which leads", and the word 'consequent' means "that which follows". In a conditional, the first constituent is called the antecedent, and the second constituent is called the consequent. When a conditional is stated in standard form in English, it is easy to identify the antecedent and the consequent, according to the following rule.

> 'if' introduces the antecedent
>
> 'then' introduces the consequent

The fact that the antecedent and consequent do not play equivalent roles is related to the fact that $\mathcal{A} \rightarrow \mathcal{C}$ is not generally equivalent to $\mathcal{C} \rightarrow \mathcal{A}$. Consider the following two conditionals.

if my car runs out of gas, **then** my car stops $R \rightarrow S$

if my car stops, **then** my car runs out of gas $S \rightarrow R$

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9. THE NON-TRUTH-FUNCTIONAL VERSION OF IF-THEN

In English, if-then is used in a variety of ways, many of which are *not* truth-functional. Consider the following conditional statements.

if I **lived** in L.A., then I **would** live in California

if I **lived** in N.Y.C., then I **would** live in California

The constituents of these two conditionals are given as follows; note that they are individually stated in the indicative mood, as required by English grammar.

L: I live in L.A. (Los Angeles)
N: I live in N.Y.C. (New York City)
C: I live in California

Now, for the author at least, all three simple statements are false. But what about the two conditionals? Well, it seems that the first one is *true*, since L.A. is entirely contained inside California (presently!). On the other hand, it seems that the second one is *false*, since N.Y.C. does not overlap California.

Thus, in the first case, two false constituents yield a *true* conditional, but in the second case, two false constituents yield a *false* conditional. It follows that the conditional connective employed in the above conditionals is *not* truth-functional.

The conditional connective employed above is customarily called the *subjunctive conditional* connective, since the constituent statements are usually stated in the subjunctive mood.

Since subjunctive conditionals are not truth-functional, they are not examined in sentential logic, at least at the introductory level. Rather, what is examined are the *truth functional conditional* connectives.

10. THE TRUTH-FUNCTIONAL VERSION OF IF-THEN

Insofar as we want to have a truth-functional conditional connective, we must construct its truth table. Of course, since not every use of 'if-then' in English is intended to be truth-functional, no truth functional connective is going to be completely plausible. Actually, the problem is to come up with a truth functional version of if-then that is even marginally plausible. Fortunately, there is such a connective.

By way of motivating the truth table for the truth-functional version of 'if-then', we consider *conditional promises* and *conditional requests*. Consider the following promise (made to the intro logic student by the intro logic instructor).

if you get a hundred on every exam, **then** I will give you an A

which may be symbolized

$H \rightarrow A$

Now suppose that the semester ends; under what circumstances has the instructor kept his/her promise. The relevant circumstances may be characterized as follows.

	H	A
case 1:	T	T
case 2:	T	F
case 3:	F	T
case 4:	F	F

The cases divide into two groups. In the first two cases, you get a hundred on every exam; the condition in question is activated; if the condition is activated, the question whether the promise is kept simply reduces to whether you do or don't get an A. In case 1, you get your A; the instructor has kept the promise. In case 2, you don't get your A, even though you got a hundred on every exam; the instructor has not kept the promise.

The remaining two cases are different. In these cases, you don't get a hundred on every exam, so the condition in question isn't activated. We have a choice now about evaluating the promise. We can say that no promise was made, so no obligation was incurred; or, we can say that a promise was made, and it was kept *by default*.

We follow the latter course, which produces the following truth table.

	H	A	H→A
case 1:	T	T	T
case 2:	T	F	F
case 3:	F	T	T
case 4:	F	F	T

Note carefully that in making the above promise, the instructor has not committed him(her)self about your grade when you don't get a hundred on every exam. It is a very simple promise, by itself, and may be combined with other promises. For example, the instructor has not promised *not* to give you an A if you do *not* get a hundred on every exam. Presumably, there are other ways to get an A; for example, a 99% average should also earn an A.

On the basis of these considerations, we propose the following truth table for the arrow connective, which represents the truth-functional version of ‘if-then’.

$\mathcal{A}$	$\mathcal{C}$	$\mathcal{A}\to\mathcal{C}$
T	T	T
T	F	F
F	T	T
F	F	T

$\mathcal{A}$	$\to$	$\mathcal{C}$
T	T	T
T	F	F
F	T	T
F	T	F

The information conveyed in the above tables may be summarized by either of the following statements.

> A conditional $\mathcal{A}\to\mathcal{C}$ is false
> if and only if
> the antecedent $\mathcal{A}$ is true
> and the consequent $\mathcal{C}$ is false.

> A conditional $\mathcal{A}\to\mathcal{C}$ is false
> if the antecedent $\mathcal{A}$ is true
> and the consequent $\mathcal{C}$ is false;
> otherwise, it is true.

11. THE BICONDITIONAL

We have now examined four truth-functional connectives, three of which are two-place connectives (conjunction, disjunction, conditional), and one of which is a one-place connective (negation). There is one remaining connective that is generally studied in sentential logic, the biconditional, which corresponds to the English

______________if and only if _______________

Like the conditional, the biconditional is a two-place connective; if we fill the two blanks with statements, the resulting expression is also a statement. For example, we can begin with the statements

I am happy
I am relaxed

and form the compound statement

I am happy **if and only if** I am relaxed

The symbol for the biconditional connective is '↔', which is called *double arrow*. The above compound can accordingly be symbolized thus.

$H \leftrightarrow R$

$H{\leftrightarrow}R$ is called the *biconditional* of H and R, which are individually called *constituents*. The truth table for ↔ is quite simple. One can understand a biconditional $\mathcal{A}{\leftrightarrow}\mathcal{B}$ as saying that the two constituents are equal in truth value; accordingly, $\mathcal{A}{\leftrightarrow}\mathcal{B}$ is true if $\mathcal{A}$ and $\mathcal{B}$ have the same truth value, and is false if they don't have the same truth value. This is summarized in the following tables.

$\mathcal{A}$	$\mathcal{B}$	$\mathcal{A}{\leftrightarrow}\mathcal{B}$
T	T	T
T	F	F
F	T	F
F	F	T

$\mathcal{A}$	$\leftrightarrow$	$\mathcal{B}$
T	T	T
T	F	F
F	F	T
F	T	F

The information conveyed in the above tables may be summarized by any of the following statements.

A biconditional $\mathcal{A}{\leftrightarrow}\mathcal{B}$ is true
if and only if
the constituents $\mathcal{A}$, $\mathcal{B}$ have the same truth value.

A biconditional $\mathcal{A}{\leftrightarrow}\mathcal{B}$ is false
if and only if
the constituents $\mathcal{A}$, $\mathcal{B}$ have opposite truth values.

A biconditional $\mathcal{A}{\leftrightarrow}\mathcal{B}$ is true
if its constituents have the same truth value; otherwise,
it is false.

A biconditional $\mathcal{A}\leftrightarrow\mathcal{B}$ is false
if its constituents have opposite truth values; otherwise,
it is true.

12. COMPLEX FORMULAS

As noted in Section 2, a statement connective forms larger (compound) statements out of smaller statements. Now, these smaller statements may themselves be compound statements; that is, they may be constructed out of smaller statements by the application of one or more statement connectives. We have already seen examples of this in Section 2.

Associated with each statement (simple or compound) is a symbolic abbreviation, or translation. Each acceptable symbolic abbreviation is what is customarily called a ***formula***. Basically, a formula is simply a string of symbols that is grammatically acceptable. Any ungrammatical string of symbols is not a formula.

For example, the following strings of symbols are not formulas in sentential logic; they are ungrammatical.

(n1) &∨P(Q
(n2) P&∨Q
(n3) P(∨Q(
(n4))(P&Q

By contrast, the following strings count as formulas in sentential logic.

(f1) (P & Q)
(f2) (~(P & Q) ∨ R)
(f3) ~(P & Q)
(f4) (~(P & Q) ∨ (P & R))
(f5) ~((P & Q) ∨ (P & R))

In order to distinguish grammatical from ungrammatical strings, we provide the following formal definition of formula in sentential logic. In this definition, the script letters stand for strings of symbols. The definition tells us which strings of symbols are formulas of sentential logic, and which strings are not.

(1) any upper case Roman letter is a formula;
(2) if $\mathcal{A}$ is a formula, then so is $\sim\mathcal{A}$;
(3) if $\mathcal{A}$ and $\mathcal{B}$ are formulas, then so is $(\mathcal{A} \mathbin{\&} \mathcal{B})$;
(4) if $\mathcal{A}$ and $\mathcal{B}$ are formulas, then so is $(\mathcal{A} \vee \mathcal{B})$;
(5) if $\mathcal{A}$ and $\mathcal{B}$ are formulas, then so is $(\mathcal{A} \rightarrow \mathcal{B})$;
(6) if $\mathcal{A}$ and $\mathcal{B}$ are formulas, then so is $(\mathcal{A} \leftrightarrow \mathcal{B})$;
(7) nothing else is a formula.

Let us do some examples of this definition. By clause 1, both P and Q are formulas, so by clause 2, the following are both formulas.

$\sim P \quad \sim Q$

So by clause 3, the following are all formulas.

$(P \,\&\, Q) \quad (P \,\&\, \sim Q) \quad (\sim P \,\&\, Q) \quad (\sim P \,\&\, \sim Q)$

Similarly, by clause 4, the following expressions are all formulas.

$(P \vee Q) \quad (P \vee \sim Q) \quad (\sim P \vee Q) \quad (\sim P \vee \sim Q)$

We can now apply clause 2 again, thus obtaining the following formulas.

$\sim(P \,\&\, Q) \quad \sim(P \,\&\, \sim Q) \quad \sim(\sim P \,\&\, Q) \quad \sim(\sim P \,\&\, \sim Q)$

$\sim(P \vee Q) \quad \sim(P \vee \sim Q) \quad \sim(\sim P \vee Q) \quad \sim(\sim P \vee \sim Q)$

We can now apply clause 3 to any pair of these formulas, thus obtaining the following *among others*.

$((P \vee Q) \,\&\, (P \vee \sim Q)) \qquad ((P \vee Q) \,\&\, \sim(P \vee \sim Q))$

The process described here can go on *indefinitely*. There is no limit to how long a formula can be, although most formulas are too long for humans to write.

In addition to formulas, in the strict sense, given in the above definition, there are also formulas in a less strict sense. We call these strings *unofficial formulas*. Basically, an unofficial formula is a string of symbols that is obtained from an official formula by dropping the outermost parentheses. This applies only to official formulas that have outermost parenthesis; negations do not have outer parentheses. The following is the official definition of an unofficial formula.

> An **unofficial formula** is any string of symbols that is obtained from an official formula by removing its outermost parentheses (if such exist).

We have already seen numerous examples of unofficial formulas in this chapter. For example, we symbolized the sentence

it is raining and it is sleeting

by the expression

R & S

Officially, the latter is not a formula; however, it is an unofficial formula.

The following represent the rough guidelines for dealing with unofficial formulas in sentential logic.

> When a formula stands by itself, one is permitted to drop its outermost parentheses (if such exist), thus obtaining an unofficial formula. However, an unofficial formula cannot be used to form a compound formula. In order to form a compound, one must restore the outermost parentheses, thereby converting the unofficial formula into an official formula.

Thus, the expression ‘R & S’, which is an unofficial formula, can be used to symbolize ‘it is raining and it is sleeting’. On the other hand, if we wish to symbolize the denial of this statement, which is ‘it is not both raining and sleeting’, then we must first restore the outermost parentheses, and then prefix the resulting expression by ‘∼’. This is summarized as follows.

it is raining and it is sleeting:	R & S
it is not both raining and sleeting:	∼(R & S)

13. TRUTH TABLES FOR COMPLEX FORMULAS

There are infinitely many formulas in sentential logic. Nevertheless, no matter how complex a given formula $\mathcal{A}$ is, we can compute its truth value, provided we know the truth values of its constituent atomic formulas. This is because all the connectives used in constructing $\mathcal{A}$ are *truth-functional*. In order to ascertain the truth value of $\mathcal{A}$, we simply compute it starting with the truth values of the atoms, using the truth function tables.

In this respect, at least, sentential logic is exactly like arithmetic. In arithmetic, if we know the *numerical values* assigned to the variables x, y, z, we can routinely calculate the numerical value of any compound arithmetical expression involving these variables. For example, if we know the numerical values of x, y, z, then we can compute the numerical value of $((x+y)\times z)+((x+y)\times(x+z))$. This computation is particularly simple if we have a hand calculator (provided that we know how to enter the numbers in the correct order; some calculators even solve this problem for us).

The only significant difference between sentential logic and arithmetic is that, whereas arithmetic concerns *numerical values* (1,2,3...) and *numerical functions* (+,×, etc.), sentential logic concerns *truth values* (T, F) and truth functions (&, ∨, etc.). Otherwise, the computational process is completely analogous. In particular, one builds up a complex computation on the basis of simple computations, and each simple computation is based on a table (in the case of arithmetic, the tables are stored in calculators, which perform the simple computations).

Let us begin with a simple example of computing the truth value of a complex formula on the basis of the truth values of its atomic constituents. The example we consider is the negation of the conjunction of two simple formulas P and Q, which is the formula ∼(P&Q). Now suppose that we substitute T for both P and Q; then we obtain the following expression: ∼(T&T). But we know that T&T = T, so ∼(T&T) = ∼T, but we also know that ∼T = F, so ∼(T&T) = F; this ends our computation. We can also substitute T for P and F for Q, in which case we have ∼(T&F). We know that T&F is F, so ∼(T&F) is ∼F, but ∼F is T, so ∼(T&F) is T. There are two other cases: substituting F for P and T for Q, and substituting F for both P and Q. They are computed just like the first two cases. We simply build up the larger computation on the basis of smaller computations.

These computations may be summarized in the following statements.

case 1: ∼(T&T) = ∼T = F
case 2: ∼(T&F) = ∼F = T
case 3: ∼(F&T) = ∼F = T
case 4: ∼(F&F) = ∼F = T

Another way to convey this information is in the following table.

Table 1

	P	Q	P&Q	∼(P&Q)
case 1	T	T	T	F
case 2	T	F	F	T
case 3	F	T	F	T
case 4	F	F	F	T

This table shows the computations step by step. The first two columns are the initial input values for P and Q; the third column is the computation of the truth value of the conjunction (P&Q); the fourth column is the computation of the truth value of the negation ∼(P&Q), which uses the third column as input.

Let us consider another simple example of computing the truth value of a complex formula. The formula we consider is a disjunction of (P&Q) and ∼P, that is, it is the formula (P&Q)∨∼P. As in the previous case, there are just two letters, so there are four combinations of truth values that can be substituted. The computations are compiled as follows, followed by the corresponding table.

case 1: (T&T) ∨ ∼T =
T ∨ F = T

case 2: (T&F) ∨ ∼T =
F ∨ F = F

case 3: (F&T) ∨ ∼F =
F ∨ T = T

case 4: (F&F) ∨ ∼F =
F ∨ T = T

By way of explanation, in case 1, the value of T&T is placed below the &, and the value of ~T is placed below the ~. These values in turn are combined by the ∨.

Table 2

	P	Q	P&Q	~P	(P&Q)∨~P
case 1	T	T	T	F	T
case 2	T	F	F	F	F
case 3	F	T	F	T	T
case 4	F	F	F	T	T

Let's now consider the formula that is obtained by conjoining the first formula (Table 1) with the second case formula (Table 2); the resulting formula is: ~(P&Q)&((P&Q)∨~P). Notice that the parentheses have been restored on the second formula before it was conjoined with the first formula. This formula has just two atomic formulas - P and Q - so there are just four cases to consider. The best way to compute the truth value of this large formula is simply to take the output columns of Tables 1 and 2 and combine them according to the conjunction truth table.

Table 3

	~(P&Q)	(P&Q)∨~P	~(P&Q)&((P&Q)∨~P)
case 1	F	T	F
case 2	T	F	F
case 3	T	T	T
case 4	T	T	T

In case 1, for example, the truth value of ~(P&Q) is F, and the truth value of (P&Q) ∨ ~P is T, so the value of their conjunction is F&T, which is F. If we were to construct the table for the complex formula from scratch, we would basically combine Tables 1 and 2. Table 3 represents the last three columns of such a table.

It might be helpful to see the computation of the truth value for ~(P&Q)&((P&Q)∨~P) done in complete detail for the first case. To begin with, we write down the formula, and we then substitute in the truth values for the first case. This yields the following.

~(P & Q) & ((P & Q) ∨ ~P)

case 1: ~(T & T) & ((T & T) ∨ ~T)

The first computation is to calculate T&T, which is T, so that yields

~T & (T ∨ ~T)

The next step is to calculate ~T, which is F, so this yields.

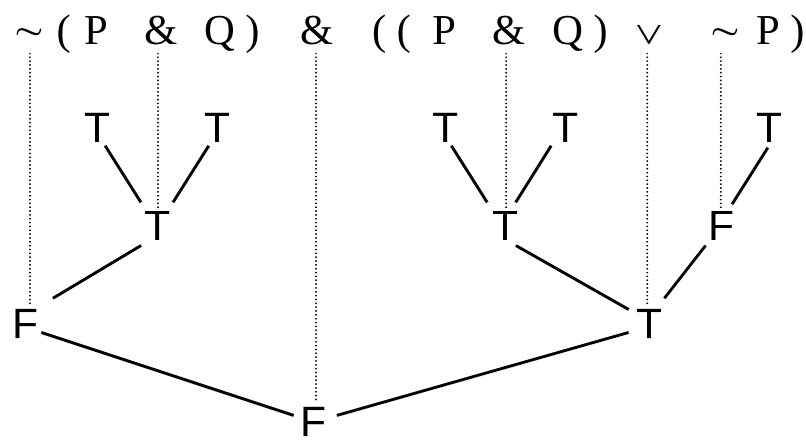
F & (T ∨ F)

Next, we calculate T ∨ F, which is T, which yields.

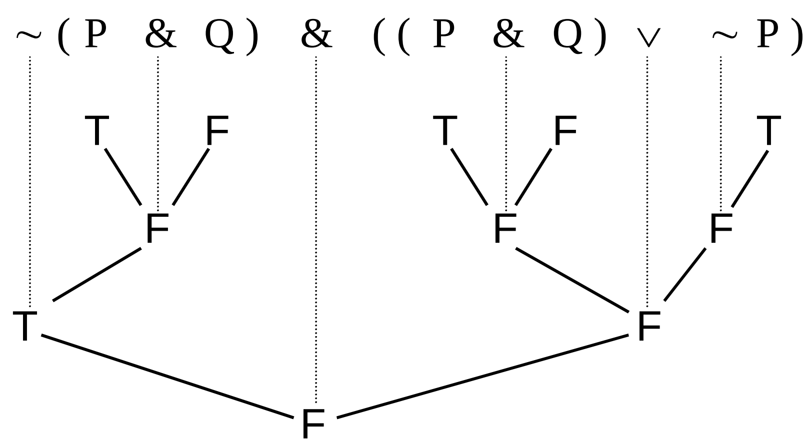
F & T

Finally, we calculate F&T, which is F, the final result in the computation.

This particular computation can be diagrammed as follows.



Case 2 can also be done in a similar manner, shown as follows.



In the above diagrams, the broken lines indicate, in each simple computation, which truth function (connective) is employed, and the solid lines indicate the input values.

In principle, in each complex computation involving truth functions, one can construct a diagram like those above for each case. Unfortunately, however, this takes up a lot of space and time, so it is helpful to have a more compact method of presenting such computations. The method that I propose simply involves superimposing all the lines above into a single line, so that each case can be presented on a single line. This can be illustrated with reference to the formulas we have already discussed.

In the case of the first formula, presented in Table 1, we can present its truth table as follows.

Table 3

	~	(	P	&	Q	)
case 1	F		T	T	T	
case 2	T		T	F	F	
case 3	T		F	F	T	
case 4	T		F	F	F	

In this table, the truth values pertaining to each connective are placed beneath that connective. Thus, for example, in case 1, the first column is the truth value of ∼(P&Q), and the third column is the truth value of (P&Q).

We can do the same with Table 2, which yields the following table.

Table 4

	(	P	&	Q	)	∨	∼	P
case 1		T	T	T		T	F	T
case 2		T	F	F		F	F	T
case 3		F	F	T		T	T	F
case 4		F	F	F		T	T	F

In this table, the second column is the truth value of (P&Q), the fourth column is the truth value of the whole formula (P&Q)∨∼P, and the fifth column is the truth value of ∼P.

Finally, we can do the compact truth table for the conjunction of the formulas given in Tables 3 and 4.

Table 5

	∼	(	P	&	Q	)	&	((	P	&	Q	)	∨	∼	P	)
case 1:	F		T	T	T		F		T	T	T		T	F	T	
case 2:	T		T	F	F		F		T	F	F		F	F	T	
case 3:	T		F	F	T		T		F	F	T		T	T	F	
case 4:	T		F	F	F		T		F	F	F		T	T	F	
	4			3			5			1			3	2		

The numbers at the bottom of the table indicate the order in which the columns are filled in. In the case of ties, this means that the order is irrelevant to the construction of the table.

In constructing compact truth tables, or in computing complex formulas, the following rules are useful to remember.

DO CONNECTIVES THAT ARE DEEPER BEFORE
DOING CONNECTIVES THAT ARE LESS DEEP.

Here, the depth of a connective is determined by how many pairs of parentheses it is inside; a connective that is inside two pairs of parentheses is deeper than one that is inside of just one pair.

AT ANY PARTICULAR DEPTH,
ALWAYS DO NEGATIONS FIRST.

These rules are applied in the above table, as indicated by the numbers at the bottom.

Before concluding this section, let us do an example of a formula that contains three atomic formulas P, Q, R. In this case, there are 8 combinations of truth values that can be assigned to the letters. These combinations are given in the following guide table.

Guide Table for any Formula Involving 3 Atomic Formulas

	P	Q	R
case 1	T	T	T
case 2	T	T	F
case 3	T	F	T
case 4	T	F	F
case 5	F	T	T
case 6	F	T	F
case 7	F	F	T
case 8	F	F	F

There are numerous ways of writing down all the combinations of truth values; this is just one particular one. The basic rule in constructing this guide table is that the rightmost column (R) is alternated T and F singly, the middle column (Q) is alternated T and F in doublets, and the leftmost column (P) is alternated T and F in quadruplets. It is simply a way of remembering all the cases.

Now let's consider a formula involving three letters P, Q, R, and its associated (compact) truth table.

Table 6

			1		2	3	4	5		6		7	8	9	10	
P	Q	R	$\sim$	[(	P	&	$\sim$	Q	)	$\vee$	(	$\sim$	P	$\vee$	R	)]
T	T	T	F		T	F	F	T		T		F	T	T	T	
T	T	F	T		T	F	F	T		F		F	T	F	F	
T	F	T	F		T	T	T	F		T		F	T	T	T	
T	F	F	F		T	T	T	F		T		F	T	F	F	
F	T	T	F		F	F	F	T		T		T	F	T	T	
F	T	F	F		F	F	F	T		T		T	F	T	F	
F	F	T	F		F	F	T	F		T		T	F	T	T	
F	F	F	F		F	F	T	F		T		T	F	T	F	
			5		1	3	2	1		4		2	1	3	1	

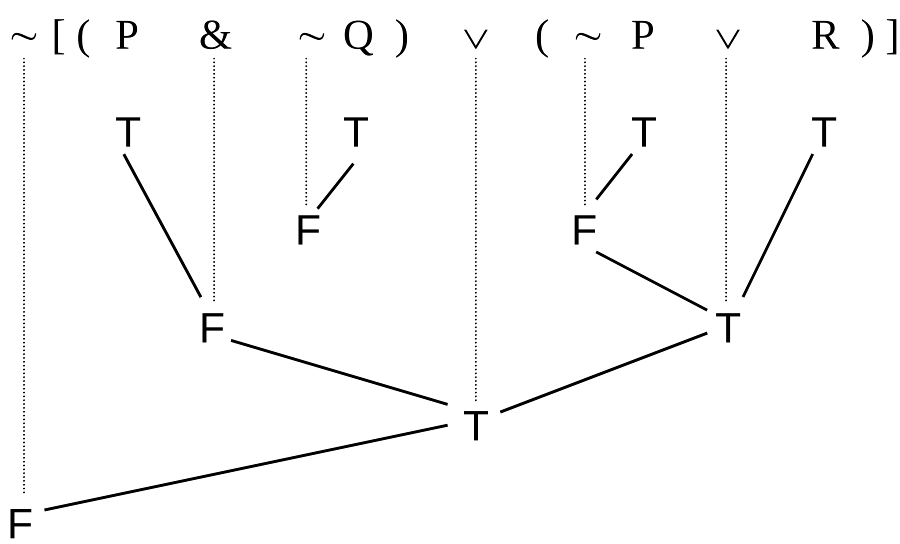
The guide table is not required, but is convenient, and is filled in first. The remaining columns, numbered 1-10 at the top, completed in the order indicated at the bottom. In the case of ties, the order doesn't matter.

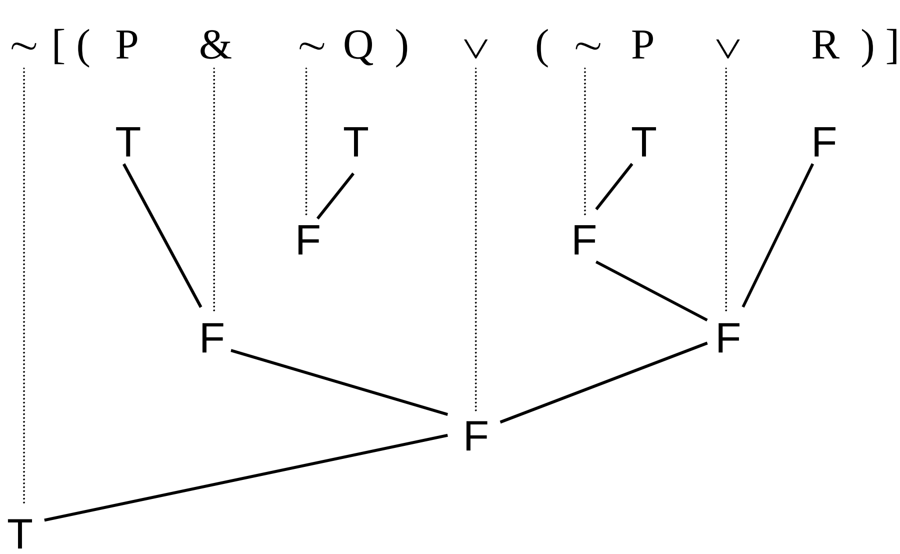
In filling a truth table, it is best to understand the structure of the formula. In case of the above formula, it is a *negation*; in particular it is the negation of the formula (P&~Q)∨(~P∨R). This formula is a *disjunction*, where the individual disjuncts are P&~Q and P∨R respectively. The first disjunct P&~Q is a conjunction of P and the negation of Q; the second disjunct ~P∨R is a disjunction of ~P and R.

The structure of the formula is crucial, and is intimately related to the order in which the truth table is filled in. In particular, the order in which the table is filled in is exactly opposite from the order in which the formula is broken into its constituent parts, as we have just done.

In filling in the above table, the first thing we do is fill in three columns under the letters, which are the smallest parts; these are labeled 1 at the bottom. Next, we do the negations of letters, which corresponds to columns 4 and 7, but *not* column 1. Column 4 is constructed from column 5 on the basis of the tilde truth table, and column 7 is constructed from column 8 in a like manner. Next column 3 is constructed from columns 2 and 4 according to the ampersand truth table, and column 9 is constructed from columns 7 and 10 according to the wedge truth table. These two resulting columns, 3 and 9, in turn go into constructing column 6 according to the wedge truth table. Finally, column 6 is used to construct column 1 in accordance with the negation truth table.

The first two cases are diagrammed in greater detail below.





As in our previous example, the broken lines indicate which truth function is applied, and the solid lines indicate the particular input values, and output values.

14. EXERCISES FOR CHAPTER 2

EXERCISE SET A

Compute the truth values of the following symbolic statements, supposing that the truth value of A, B, C is T, and the truth value of X, Y, Z is F.

1. $\sim A \vee B$
2. $\sim B \vee X$
3. $\sim Y \vee C$
4. $\sim Z \vee X$
5. $(A \& X) \vee (B \& Y)$
6. $(B \& C) \vee (Y \& Z)$
7. $\sim(C \& Y) \vee (A \& Z)$
8. $\sim(A \& B) \vee (X \& Y)$
9. $\sim(X \& Z) \vee (B \& C)$
10. $\sim(X \& \sim Y) \vee (B \& \sim C)$
11. $(A \vee X) \& (Y \vee B)$
12. $(B \vee C) \& (Y \vee Z)$
13. $(X \vee Y) \& (X \vee Z)$
14. $\sim(A \vee Y) \& (B \vee X)$
15. $\sim(X \vee Z) \& (\sim X \vee Z)$
16. $\sim(A \vee C) \vee \sim(X \& \sim Y)$
17. $\sim(B \vee Z) \& \sim(X \vee \sim Y)$
18. $\sim[(A \vee \sim C) \vee (C \vee \sim A)]$
19. $\sim[(B \& C) \& \sim(C \& B)]$
20. $\sim[(A \& B) \vee \sim(B \& A)]$
21. $[A \vee (B \vee C)] \& \sim[(A \vee B) \vee C]$
22. $[X \vee (Y \& Z)] \vee \sim[(X \vee Y) \& (X \vee Z)]$
23. $[A \& (B \vee C)] \& \sim[(A \& B) \vee (A \& C)]$
24. $\sim\{[(\sim A \& B) \& (\sim X \& Z)] \& \sim[(A \& \sim B) \vee \sim(\sim Y \& \sim Z)]\}$
25. $\sim\{\sim[(B \& \sim C) \vee (Y \& \sim Z)] \& [(\sim B \vee X) \vee (B \vee \sim Y)]\}$

EXERCISE SET B

Compute the truth values of the following symbolic statements, supposing that the truth value of A, B, C is T, and the truth value of X, Y, Z is F.

1. $A \rightarrow B$
2. $A \rightarrow X$
3. $B \rightarrow Y$
4. $Y \rightarrow Z$
5. $(A \rightarrow B) \rightarrow Z$
6. $(X \rightarrow Y) \rightarrow Z$
7. $(A \rightarrow B) \rightarrow C$
8. $(X \rightarrow Y) \rightarrow C$
9. $A \rightarrow (B \rightarrow Z)$
10. $X \rightarrow (Y \rightarrow Z)$
11. $[(A \rightarrow B) \rightarrow C] \rightarrow Z$
12. $[(A \rightarrow X) \rightarrow Y] \rightarrow Z$
13. $[A \rightarrow (X \rightarrow Y)] \rightarrow C$
14. $[A \rightarrow (B \rightarrow Y)] \rightarrow X$
15. $[(X \rightarrow Z) \rightarrow C] \rightarrow Y$
16. $[(Y \rightarrow B) \rightarrow Y] \rightarrow Y$
17. $[(A \rightarrow Y) \rightarrow B] \rightarrow Z$
18. $[(A \,\&\, X) \rightarrow C] \rightarrow [(X \rightarrow C) \rightarrow X]$
19. $[(A \,\&\, X) \rightarrow C] \rightarrow [(A \rightarrow X) \rightarrow C]$
20. $[(A \,\&\, X) \rightarrow Y] \rightarrow [(X \rightarrow A) \rightarrow (A \rightarrow Y)]$
21. $[(A \,\&\, X) \vee (\sim A \,\&\, \sim X)] \rightarrow [(A \rightarrow X) \,\&\, (X \rightarrow A)]$
22. $\{[A \rightarrow (B \rightarrow C)] \rightarrow [(A \,\&\, B) \rightarrow C]\} \rightarrow [(Y \rightarrow B) \rightarrow (C \rightarrow Z)]$
23. $\{[(X \rightarrow Y) \rightarrow Z] \rightarrow [Z \rightarrow (X \rightarrow Y)]\} \rightarrow [(X \rightarrow Z) \rightarrow Y]$
24. $[(A \,\&\, X) \rightarrow Y] \rightarrow [(A \rightarrow X) \,\&\, (A \rightarrow Y)]$
25. $[A \rightarrow (X \,\&\, Y)] \rightarrow [(A \rightarrow X) \vee (A \rightarrow Y)]$

EXERCISE SET C

Construct the complete truth table for each of the following formulas.

1. $(P \,\&\, Q) \vee (P \,\&\, {\sim}Q)$
2. ${\sim}(P \,\&\, {\sim}P)$
3. ${\sim}(P \vee {\sim}P)$
4. ${\sim}(P\&Q)\vee({\sim}P\vee{\sim}Q)$
5. ${\sim}(P \vee Q) \vee ({\sim}P \,\&\, {\sim}Q)$
6. $(P \,\&\, Q) \vee ({\sim}P \,\&\, {\sim}Q)$
7. ${\sim}(P \vee (P \,\&\, Q))$
8. ${\sim}(P \vee (P \,\&\, Q)) \vee P$
9. $(P \,\&\, (Q \vee P)) \,\&\, {\sim}P$
10. $((P \rightarrow Q) \rightarrow P) \rightarrow P$
11. ${\sim}({\sim}(P \rightarrow Q) \rightarrow P)$
12. $(P \rightarrow Q) \leftrightarrow {\sim}P$
13. $P \rightarrow (Q \rightarrow (P \,\&\, Q))$
14. $(P \vee Q) \leftrightarrow ({\sim}P \rightarrow Q)$
15. ${\sim}(P \vee (P \rightarrow Q))$
16. $(P \rightarrow Q) \leftrightarrow (Q \rightarrow P)$
17. $(P \rightarrow Q) \leftrightarrow ({\sim}Q \rightarrow {\sim}P)$
18. $(P \vee Q) \rightarrow (P \,\&\, Q)$
19. $(P \,\&\, Q) \vee (P \,\&\, R)$
20. $[P \leftrightarrow (Q \leftrightarrow R)] \leftrightarrow [(P \leftrightarrow Q) \leftrightarrow R]$
21. $[P \rightarrow (Q \,\&\, R)] \rightarrow [P \rightarrow R]$
22. $[P \rightarrow (Q \vee R)] \rightarrow [P \rightarrow Q]$
23. $[(P \vee Q) \rightarrow R] \rightarrow [P \rightarrow R]$
24. $[(P \,\&\, Q) \rightarrow R] \rightarrow [P \rightarrow R]$
25. $[(P \,\&\, Q) \rightarrow R] \rightarrow [(Q \,\&\, {\sim}R) \rightarrow {\sim}P]$

15. ANSWERS TO EXERCISES FOR CHAPTER 2

EXERCISE SET A

1. T
2. F
3. T
4. T
5. F
6. F
7. T
8. F
9. T
10. T
11. T
12. F
13. F
14. F
15. T
16. T
17. F
18. F
19. T
20. F
21. F
22. T
23. F
24. T
25. F

EXERCISE SET B

1. T
2. F
3. F
4. T
5. F
6. T
7. T
8. T
9. F
10. T
11. F
12. F
13. T
14. T
15. F
16. T
17. F
18. F
19. T
20. F
21. T
22. F
23. F
24. F
25. T

EXERCISE SET C

1.

(	P	&	Q	)	∨	(	P	&	~	Q	)
	T	T	T		T		T	F	F	T	
	T	F	F		T		T	T	T	F	
	F	F	T		F		F	F	F	T	
	F	F	F		F		F	F	T	F	

2.

~	(	P	&	~	P	)
T		T	F	F	T	
T		F	F	T	F	

3.

~	(	P	∨	~	P	)
F		T	T	F	T	
F		F	T	T	F	

4.

~	(	P	&	Q	)	∨	(	~	P	∨	~	Q	)
F		T	T	T		F		F	T	F	F	T	
T		T	F	F		T		F	T	T	T	F	
T		F	F	T		T		T	F	T	F	T	
T		F	F	F		T		T	F	T	T	F	

5.

~	(	P	∨	Q	)	∨	(	~	P	&	~	Q	)
F		T	T	T		F		F	T	F	F	T	
F		T	T	F		F		F	T	F	T	F	
F		F	T	T		F		T	F	F	F	T	
T		F	F	F		T		T	F	T	T	F	

6.

(	P	&	Q	)	∨	(	~	P	&	~	Q	)
	T	T	T		T		F	T	F	F	T	
	T	F	F		F		F	T	F	T	F	
	F	F	T		F		T	F	F	F	T	
	F	F	F		T		T	F	T	T	F	

7.

~	(	P	∨	(	P	&	Q	)	)
F		T	T		T	T	T		
F		T	T		T	F	F		
T		F	F		F	F	T		
T		F	F		F	F	F		

8.

~	(	P	∨	(	P	&	Q	)	)	∨	P
F		T	T		T	T	T			T	T
F		T	T		T	F	F			T	T
T		F	F		F	F	T			T	F
T		F	F		F	F	F			T	F

9.

(	P	&	(	Q	∨	P	)	)	&	~	P
	T	T		T	T	T			F	F	T
	T	T		F	T	T			F	F	T
	F	F		T	T	F			F	T	F
	F	F		F	F	F			F	T	F

10.

(	(	P	→	Q	)	→	P	)	→	P
		T	T	T		T	T		T	T
		T	F	F		T	T		T	T
		F	T	T		F	F		T	F
		F	T	F		F	F		T	F

11.

~	(	~	(	P	→	Q	)	→	P	)
F		F		T	T	T		T	T	
F		T		T	F	F		T	T	
F		F		F	T	T		T	F	
F		F		F	T	F		T	F	

12.

(	P	→	Q	)	↔	~	P
	T	T	T		F	F	T
	T	F	F		T	F	T
	F	T	T		T	T	F
	F	T	F		T	T	F

13.

P	→	(	Q	→	(	P	&	Q	)	)
T	T		T	T		T	T	T		
T	T		F	T		T	F	F		
F	T		T	F		F	F	T		
F	T		F	T		F	F	F		

14.

(	P	∨	Q	)	↔	(	~	P	→	Q	)
	T	T	T		T		F	T	T	T	
	T	T	F		T		F	T	T	F	
	F	T	T		T		T	F	T	T	
	F	F	F		T		T	F	F	F	

15.

~	(	P	∨	(	P	→	Q	)	)
F		T	T		T	T	T		
F		T	T		T	F	F		
F		F	T		F	T	T		
F		F	T		F	T	F		

16

(	P	→	Q	)	↔	(	Q	→	P	)
	T	T	T		T		T	T	T	
	T	F	F		F		F	T	T	
	F	T	T		F		T	F	F	
	F	T	F		T		F	T	F	

17.

(	P	→	Q	)	↔	(	~	Q	→	~	P	)
	T	T	T		T		F	T	T	F	T	
	T	F	F		T		T	F	F	F	T	
	F	T	T		T		F	T	T	T	F	
	F	T	F		T		T	F	T	T	F	

18.

(	P	∨	Q	)	→	(	P	&	Q	)
	T	T	T		T		T	T	T	
	T	T	F		F		T	F	F	
	F	T	T		F		F	F	T	
	F	F	F		T		F	F	F	

19.

(	P	&	Q	)	∨	(	P	&	R	)
	T	T	T		T		T	T	T	
	T	T	T		T		T	F	F	
	T	F	F		T		T	T	T	
	T	F	F		F		T	F	F	
	F	F	T		F		F	F	T	
	F	F	T		F		F	F	F	
	F	F	F		F		F	F	T	
	F	F	F		F		F	F	F	

20.

[P	$\leftrightarrow$	(Q	$\leftrightarrow$	R)]	$\leftrightarrow$	[(P	$\leftrightarrow$	Q)	$\leftrightarrow$	R]
T	T	T	T	T	T	T	T	T	T	T
T	F	T	F	F	T	T	T	T	F	F
T	F	F	F	T	T	T	F	F	F	T
T	T	F	T	F	T	T	F	F	T	F
F	F	T	T	T	T	F	F	T	F	T
F	T	T	F	F	T	F	F	T	T	F
F	T	F	F	T	T	F	T	F	T	T
F	F	F	T	F	T	F	T	F	F	F

21.

[P	$\rightarrow$	(Q	&	R)]	$\rightarrow$	[P	$\rightarrow$	R]
T	T	T	T	T	T	T	T	T
T	F	T	F	F	T	T	F	F
T	F	F	F	T	T	T	T	T
T	F	F	F	F	T	T	F	F
F	T	T	T	T	T	F	T	T
F	T	T	F	F	T	F	T	F
F	T	F	F	T	T	F	T	T
F	T	F	F	F	T	F	T	F

22.

[P	$\rightarrow$	(Q	$\vee$	R)]	$\rightarrow$	[P	$\rightarrow$	Q]
T	T	T	T	T	T	T	T	T
T	T	T	T	F	T	T	T	T
T	T	F	T	T	F	T	F	F
T	F	F	F	F	T	T	F	F
F	T	T	T	T	T	F	T	T
F	T	T	T	F	T	F	T	T
F	T	F	T	T	T	F	T	F
F	T	F	F	F	T	F	T	F

23.

[(P	$\vee$	Q)	$\rightarrow$	R]	$\rightarrow$	[P	$\rightarrow$	R]
T	T	T	T	T	T	T	T	T
T	T	T	F	F	T	T	F	F
T	T	F	T	T	T	T	T	T
T	T	F	F	F	T	T	F	F
F	T	T	T	T	T	F	T	T
F	T	T	F	F	T	F	T	F
F	F	F	T	T	T	F	T	T
F	F	F	T	F	T	F	T	F

24.

[	(	P	&	Q	)	→	R	]	→	[	P	→	R	]
		T	T	T		T	T		T		T	T	T	
		T	T	T		F	F		T		T	F	F	
		T	F	F		T	T		T		T	T	T	
		T	F	F		T	F		F		T	F	F	
		F	F	T		T	T		T		F	T	T	
		F	F	T		T	F		T		F	T	F	
		F	F	F		T	T		T		F	T	T	
		F	F	F		T	F		T		F	T	F	

25.

[	(	P	&	Q	)	→	R	]	→	[	(	Q	&	~	R	)	→	~	P	]
		T	T	T		T	T		T			T	F	F	T		T	F	T	
		T	T	T		F	F		T			T	T	T	F		F	F	T	
		T	F	F		T	T		T			F	F	F	T		T	F	T	
		T	F	F		T	F		T			F	F	T	F		T	F	T	
		F	F	T		T	T		T			T	F	F	T		T	T	F	
		F	F	T		T	F		T			T	T	T	F		T	T	F	
		F	F	F		T	T		T			F	F	F	T		T	T	F	
		F	F	F		T	F		T			F	F	T	F		T	T	F	