(a) Either a face card (jack, queen, king) or a ten?
(b) Either a spade or a face card?

3 *Two cards.* When two cards are drawn in succession from a standard pack of cards, what are the probabilities of drawing:
(a) Two hearts in a row, with replacement, and (b) without replacement.
(c) Two cards, neither of which is a heart, with replacement, and (d) without replacement.

4 *Archery.* An archer's target has four concentric circles around a bull's-eye. For a certain archer, the probabilities of scoring are as follows:

Pr(hit the bull's-eye) = 0.1
Pr(hit first circle, but not bull's-eye) = 0.3.
Pr(hit second circle, but no better) = 0.2.
Pr(hit third circle, but no better) = 0.2.
Pr(hit fourth circle, but no better) = 0.1.

Her shots are independent.
(a) What is the probability that in two shots she scores a bull's-eye on the first shot, and the third circle on the second shot?
(b) What is the probability that in two shots she hits the bull's-eye once, and the third circle once?
(c) What is the probability that on any one shot she misses the target entirely?

5 *Polio from diapers* (a news story).

> *Southampton, England:* A man contracted polio from the soiled diaper of his niece, who had been vaccinated against the disease just days before, doctors said yesterday. "The probability of a person contracting polio from soiled diapers is literally one in three million," said consultant Martin Wale. What did Dr. Wale mean?

6 *Languages.* We distinguish between a "proposition-language" and an "event-language" for probability. Which language was used in:
(a) Question 2. (b) Question 3. (c) Question 4. (d) Dr. Wale's statement in question 5?

KEY WORDS FOR REVIEW

Events	Addition
Propositions	Independence
Mutually exclusive	Multiplication

5 Conditional Probability

The most important new idea about probability is the probability that something happens, on condition that something else happens. This is called conditional probability.

CATEGORICAL AND CONDITIONAL

We express probabilities in numbers. Here is a story I read in the newspaper. The old tennis pro Ivan was discussing the probability that the rising young star Stefan would beat the established player Boris in the semifinals. Ivan was set to play Pete in the other semifinals match. He said,

> The probability that Stefan will beat Boris is 40%.

Or he could have said,

> The chance of Stefan's winning is 0.4.

These are *categorical* statements, no ifs and buts about them. Ivan might also have this opinion:

> Of course I'm going to win my semifinal match, but if I were to lose, then Stefan would not be so scared of meeting me in the finals, and he would play better; there would then be a 50–50 chance that Stefan would beat Boris.

This is the probability of Stefan's winning in his semifinal match, *conditional* on Ivan losing the other semifinal. We call it the *conditional probability*. Here are other examples:

> *Categorical*: The probability that there will be a bumper grain crop on the prairies next summer.

Conditional: The probability that there will be a bumper grain crop next summer, given that there has been very heavy snowfall the previous winter.

Categorical: The probability of dealing an ace as the second card from a standard pack of well-shuffled cards (regardless of what card is dealt first).

There are 4 aces and 52 cards, any one of which may come up as the second card. So the probability of getting an ace as the second card should be 4/52 = 1/13.

Conditional: The probability of dealing an ace as the second card *on condition that* the first card dealt was a king.

If a king was dealt first, there are 51 cards remaining. There are 4 aces still in the pack, so the *conditional probability* is 4/51.

Conditional: The probability of dealing an ace as the second card on condition that the first card dealt was also an ace.

When an ace is dealt first, there are 51 cards remaining, but only 3 aces, so the conditional probability is 3/51.

NOTATION

Categorical probability is represented:

Pr( )

Conditional probability is represented:

Pr(/).

Examples of categorical probability:

Pr(S wins the final) = 0.4.
Pr(second card dealt is an ace) = 1/13.

Examples of conditional probability:

Pr(S wins his semifinal/I loses his semifinal) = 0.5.
Pr(second card dealt is an ace/first card dealt is a king) = 4/51.

BINGO

Bingo players know about conditional probability.

In a game of bingo, you have a 5×5 card with 25 squares. Each square is marked with a different number from 1 to 99. The master of ceremonies draws numbered balls from a bag. Each time a number on your board is drawn, you fill in the corresponding square. You win (BINGO!) when you fill in a complete column, row, or diagonal.

Bingo players are fairly relaxed when they start the game. The probability that they will soon complete a line is small. But as they begin to fill in a line they get very excited, because the conditional probability of their winning is not so small.

PARKING TICKETS

If you park overnight near my home, and don't live on the block, you may be ticketed for not having a permit for overnight parking. The fine will be $20. But the street is only patrolled on average about once a week.

What is the probability of being fined?

Apparently the street is never patrolled on two consecutive nights. What is the probability of being ticketed tonight, *conditional* on having been ticketed on this street last night?

DEFINITION OF CONDITIONAL PROBABILITY

There is a very handy definition of conditional probability. We first state it, and then illustrate how it works.

When Pr(B) > 0
Pr(A/B) = Pr(A&B)/Pr(B)

Pr(B) must be a positive number, because we cannot divide by zero. But why is the rest of this definition sensible? Some examples will suggest why.

CONDITIONAL DICE

Think of a fair die. We say the outcome of a toss is even if it falls 2, 4, or 6 face up.

Here is conditional probability:

Pr(6/even)

In ordinary English:

The probability that we roll a 6, on condition that we rolled an even number.
The conditional probability of sixes, given evens.

With a fair die, we roll 2, 4, and 6 equally often. So 6 comes up a third of the time that we get an even outcome.

Pr(6/even) = 1/3.

This fits our definition, because,

Pr(6 & even) = Pr(6) = 1/6.
Pr(even) = 1/2.
Pr(6/even) = (1/6)/(1/2) = 1/3.

OVERLAPS

Now ask a more complicated question, which involves *overlapping* outcomes. Let M mean that the die either falls 1 up or falls with a prime number up (2, 3, 5). Thus M happens when the die falls 1, 2, 3, or 5 uppermost. What is

Pr(even/M)?

The only prime even number is 2. There are 4 ways to throw M (1, 2, 3, 5). Hence, if the die is fair,

Pr(even/M) = 1/4.

This fits our definition, because

Pr(even & M) = 1/6.
Pr(M) = 4/6.
Pr(even/M) = (1/6)/(4/6) = 1/4.

WELL-SHUFFLED CARDS

Think of a well-shuffled standard pack of 52 cards, from which the dealer deals the top card. He tells you that it is *either* red, *or* clubs. But not which. Call this information RvC.

Clubs are black. There are 13 clubs in the pack, and 26 other cards that are red. We were told that the first card is RvC. What is the probability that it is an ace? What is Pr(A/RvC)? A & (RvC) is equivalent to ace of clubs, or a red ace, diamonds or hearts. For a total of 3. Hence 3 cards out of the 39 RvC cards are aces.

Hence the conditional probability is:

Pr[A/(RvC)] = 1/13.

This agrees with our definition:

Pr[A & (RvC)] = 3/52.
Pr(RvC) = 39/52.

$$\Pr[(A/(RvC)] = \frac{\Pr[A \,\&\, (RvC)]}{\Pr(RvC)} = 3/39 = 1/13.$$

URNS

Imagine two urns, each containing red and green balls. Urn A has 80% red balls, 20% green, and Urn B has 60% green, 40% red. You pick an urn at random. Is it A or B? Let's draw balls from the urn and use this information to guess which urn it is. After each draw, the ball drawn *is replaced*. Hence for any draw, the probability of getting red from urn A is 0.8, and from urn B, the probability of getting red is 0.4.

Pr(R/A) = 0.8
Pr(R/B) = 0.4
Pr(A) = Pr(B) = 0.5

You draw a red ball. If you are like *Alert Learner*, that may lead you to suspect that this is urn A (which has more red balls than green ones). That is just a hunch. Let's be more exact.

We want to find Pr(A/R), which is [Pr(A&R)]/[Pr(R)].

You can get a red ball from either urn A or urn B. You get a red ball either when the event A&R happens, or when the event B&R happens. Event R is thus identical to (A&R)v(B&R).

The two alternatives (A&R) and (B&R) are mutually exclusive, so we can add up the probabilities.

Pr(R) = Pr(A&R) + Pr(B&R) [1]

The probability of getting urn B is 0.5; the probability of getting a red ball from it is 0.4, so that the probability of both happening is

Pr(B&R) = Pr(R&B) = Pr(R/B)Pr(B) = 0.4 × 0.5 = 0.2.

Likewise,

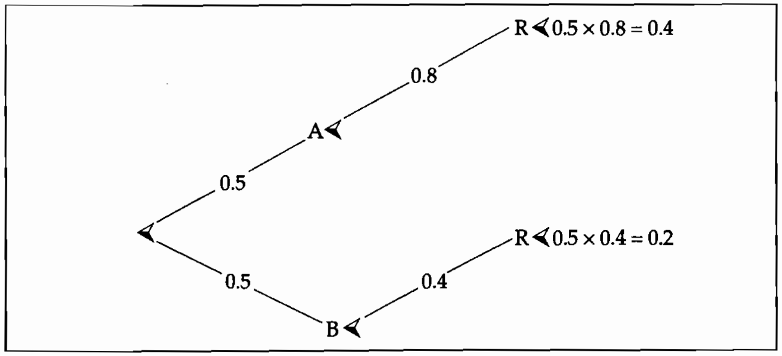
Pr(A&R) = 0.8 × 0.5 = 0.4.

Putting these into formula [1] above,

Pr(R) = Pr(A&R) + Pr(B&R) = 0.4 + 0.2 = 0.6.
Hence Pr(A/R) = Pr(A&R)/Pr(R) = (0.4)/(0.6) = 2/3.

DRAWING THE CALCULATION TO CHECK IT

You may find it helpful to visualize the calculation as a branching tree. We start out with our coin and the two urns. How can we get to a red ball? There are two routes. We can toss a heads (probability 0.5), giving us urn A. Then we can draw a red ball (probability 0.8). That is the route shown here on the top branch.



We can also get an R by tossing tails, going to urn B, and then drawing a red ball, as shown on the bottom branch.

We get to R on one of the two branches. So the total probability of ending up with R is the sum of the probabilities at the end of each branch. Here it is 0.4 + 0.2 = 0.6.

The probability of getting to an R following an A branch is 0.4.

Thus that part of the probability that gets you to R by A, namely Pr(A/R), is .4/.6 = 2/3.

MODELS

All our examples up to now have been dice, cards, urns. Now we turn to more interesting cases, more like real life. In each we make a model of a situation, and say that the real-life story is modeled by a standard ball-and-urn example.

SHOCK ABSORBERS

An automobile plant contracted to buy shock absorbers from two local suppliers, Bolt & Co. and Acme Inc. Bolt supplies 40% and Acme 60%. All shocks are subject to quality control. The ones that pass are called reliable.

Of Acme's shocks, 96% test reliable. But Bolt has been having some problems on the line, and recently only 72% of Bolt's shock absorbers have tested reliable.

What is the probability that a randomly chosen shock absorber will test reliable?

Intuitive guess: the probability will be lower than 0.96, because Acme's product is diluted by a proportion of shock absorbers from Bolt. The probability must be between 0.96 and 0.72, and nearer to 0.96. But by how much?

Solution

Let A = The shock chosen at random was made by Acme.
Let B = The shock chosen at random was made by Bolt.
Let R = The shock chosen at random is reliable.

Pr(A) = 0.6	Pr(R/A) = 0.96	So, Pr(R&A) = 0.576.
Pr(B) = 0.4	Pr(R/B) = 0.72	So, Pr(R&B) = 0.288.

R = (R&A)v(R&B)

Answer: Pr(R) = (.6×.96) + (.4×.72) = 0.576 + 0.288 = 0.864.

We can ask a more interesting question.

What is the conditional probability that a randomly chosen shock absorber, which is tested and found to be reliable, is made by Bolt?

Intuitive guess: look at the numbers. The automobile plant buys more shocks from Acme than Bolt. And Bolt's shocks are much less reliable than Acme's. Both these pieces of evidence count against a reliable shock, chosen at random, being made by Bolt. We expect that the probability that the shock is from Bolt is less that 0.4. But by how much?

Solution

We require Pr(B/R).

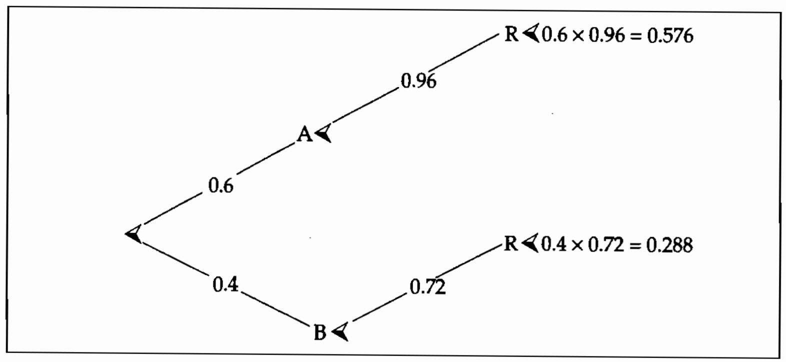
By definition, Pr(B/R) = Pr(B&R)/Pr(R) = 0.288/0.864 = 1/3.

Actually, you may like to do this without any multiplying, because almost all the numbers cancel:

$$= \frac{0.4 \times 0.72}{(.4\times.72) + (.6\times.96)} = 1/3$$

Answer: Pr(B/R) = 1/3.

DRAWING TO CHECK



Pr(R) = 0.576 + 0.288 = 0.864. Pr(B/R) = 1/3.

WEIGHTLIFTERS

You learn that a certain country has two teams of weightlifters, either of which it may send to an international competition. Of the members of one team (the Steroid team), 80% have been regularly using steroids, but only 20% of the members of the other team are regular users (the Cleaner team). The head coach flips a fair coin to decide which team will be sent.

One member of the competing team is tested at random. He has been using steroids.

What is the conditional probability that the team in competition is the Steroid team, given that a member was found by a urine test to be using steroids? That is, what is Pr(S/U)?

Solution

Let S = The coach sent the Steroid team.
Let C = The coach sent the Cleaner team.
Let U = A member selected at random uses steroids.

Pr(S) = 0.5	Pr(U/S) = 0.8	Pr(U&S) = 0.4
Pr(C) = 0.5	Pr(U/C) = 0.2	Pr(U&C) = 0.1
U = (U&S)v(U&C)	Pr(U) = 0.4 + 0.1 = 0.5	

Pr(S/U) = [Pr(S&U)]/[Pr(U)] = 0.4/0.5 = 0.8

Answer: Pr(Steroid team/selected member uses steroids) = 0.8.

So the fact that we randomly selected a team member who uses steroids, is pretty good evidence that this is the Steroid team.

TWO IN A ROW: WITH REPLACEMENT

Back to the urns on page 51. Suppose you pick an urn at random, and make two draws, *with replacement*. You get a red, and then a red again. What is $P(A/R_1\&R_2)$?

Let R_1 be the event that the first ball drawn is red, and R_2 the event that the second ball drawn is red. Then you can work out $P(A/R_1\&R_2)$ as:

$$\frac{Pr(A\&R_1\&R_2)}{Pr(R_1\&R_2)}$$

Now we know $Pr(A\&R_1\&R_2) = Pr(R_2/A\&R_1)Pr(A\&R_1) = 0.8 \times 0.4 = 0.32$.
Likewise, $Pr(B\&R_1\&R_2) = 0.08$.
$Pr(R_1\&R_2) = Pr(A\&R_1\&R_2) + Pr(B\&R_1\&R_2) = 0.32 + 0.08 = 0.4$
$Pr(A/R_1\&R_2) = 0.32/0.4 = 4/5 = 0.8$

Conditional probability of urn A, given that we:
draw one red ball, is 2/3
draw a second ball after replacement, also red, is 0.8

Thus a second red ball "increases the conditional probability" that this is urn A. The extra red ball may be taken as more *evidence*.

This suggests how we *learn by experience* by obtaining more evidence.

THE GAMBLER'S FALLACY ONCE AGAIN

Fallacious Gambler thought that he could "learn from experience" when he saw that a fair (unbiased, independent trials) roulette wheel came up 12 blacks in a row. That is, he thought that:

Pr(red on 13th trial/12 blacks in a row) > ½.

But if trials are independent, this probability is

$$\frac{Pr(BBBBBBBBBBBBR)}{Pr(BBBBBBBBBBBB)} = \frac{(1/2)^{13}}{(1/2)^{12}} = 1/2.$$

This is a new way to understand the gambler's fallacy.

TWO WEIGHTLIFTERS: WITHOUT REPLACEMENT

Back to the weightlifters. Suppose we test two weightlifters chosen at random from a team that the coach selected by tossing a fair coin. We think: if both weightlifters test positive, that is pretty strong evidence that this is the Steroid team. Probability confirms this hunch.

We are sampling the team without replacement. So say there are ten members to a team. We randomly test two members.

Let S = The coach sent the Steroid team.
Let C = The coach sent the Cleaner team.
Let U_1 = The first member selected at random uses steroids.
Let U_2 = The second member selected at random uses steroids.

If we have the Steroid team, the probability that the first person tested uses steroids is 0.8 (on page 54 we had Pr(U/S) = 0.8). What is the probability of selecting two users?

There is a 4/5 probability of selecting one user. After the first person is selected, and turns out to be a user, there are 9 team members left, 7 of whom use steroids. So there is a 7/9 probability of getting a user for the next test. Hence the probability that the first two persons chosen from the Steroid team use steroids is $4/5 \times 7/9 = 28/45$.

Likewise, the probability that the first two persons chosen from the Cleaner team use steroids is $1/5 \times 1/9 = 1/45$.

The probability that the coach sent the Steroid team, when both team members selected at random are users, is,

$\Pr(S\&U_1\&U_2) = 0.5(28/45) = 28/90.$

Likewise, $\Pr(C\&U_1\&U_2) = 0.5(1/45) = 1/90.$

$\Pr(U_1\&U_2) = \Pr(S\&U_1\&U_2) + \Pr(C\&U_1\&U_2) = 29/90.$

$$\Pr(S/U_1\&U_2) = \frac{\Pr(S\&U_1\&U_2)}{\Pr(U_1\&U_2)} = 28/29 > 0.96$$

Conditional probability that this is the Steroid team, given that we randomly selected:
one weightlifter who was a user, is 0.8
a second weightlifter who was also a user, is > 0.96

Getting two members who use steroids seems to be powerful *evidence* that the coach picked the Steroid team.

EXERCISES

1 *Phony precision about tennis.* In real life, the newspaper story about tennis quoted Ivan as stating a probability not of "40%" but:

> The probability that Stefan will beat Boris in the semifinals is only 37.325%.

Can you make any sense out of this precise fraction?

2 *Heat lamps.* Three percent of production batches of *Tropicana* heat lamps fall below quality standards. Six percent of the batches of *Florida* heat lamps are below quality standards. A hardware store buys 40% of its heat lamps from *Tropicana*, and 60% from *Florida*.
(a) What is the probability that a lamp taken at random in the store is made by *Tropicana* and is below quality standards?
(b) What is the probability that a lamp taken at random in the store is below quality standards?
(c) What is the probability that a lamp from this store, and found to be below quality standards, is made by *Tropicana*?

3 *The Triangle.* An unhealthy triangular-shaped region in an old industrial city once had a lot of chemical industry. Two percent of the children in the city live in the triangle. Fourteen percent of these test positive for excessive presence of toxic metals in the tissues. The rate of positive tests for children in the city, not living in the triangle, is only 1%.
(a) What is the probability that a child who lives in the city, and who is chosen at random, both lives in the Triangle and tests positive?
(b) What is the probability that a child living in the city, chosen at random, tests positive?
(c) What is the probability that a child chosen at random, who tests positive, lives in the Triangle?

4 *Taxicabs.* Draw a tree diagram for the taxicab problem, Odd Question 5.

5 *Laplace's trick question.* Look back at Laplace's question (page 44). An experiment consists of tossing a coin to select an urn, then drawing a ball, noting its color, replacing it, and drawing another ball and noting its color. Find, Pr(second ball drawn is red/first ball drawn is red).

6 *Understanding the question.* On page 45 we ended Chapter 4 with a way to understand Laplace's trick question. How does your answer to question 6 help with understanding Laplace's question?

KEY WORDS FOR REVIEW

Categorical
Conditional
Definition of conditional probability
Calculating conditional probabilities
Models
Learning from experience