Announcements & Overview

• Administrative Stuff
  - HW #2 grades & solutions are now posted
  - HW #3 due today (I will post solutions next week)
  - HW #4 Posted (due in 2 weeks)
  - The mid-term is next Friday — March 2
    ∗ I have posted a practice mid-term exam
    ∗ We will go over the practice mid-term on Tuesday
    ∗ I have also posted a mid-term rules handout
  - Special Office Hours next week: Wed 12–2

• Today: Natural Deduction, Continued
  - Sequent & Theorem Introduction in LSL
  - Natural deduction in LMPL/L2PL
The Rule of Definition for the Biconditional

**Rule of Definition for \( \leftrightarrow \) (Df):** If \( (p \rightarrow q) \& (q \rightarrow p) \) occurs as the entire formula at line \( j \), then at line \( k \) we may write \( p \leftrightarrow q \), labeling the line ‘\( j \text{ Df} \)’ and writing on its left the same numbers as are on the left of \( j \). Conversely, if \( p \leftrightarrow q \) occurs as the entire formula at a line \( j \), then at line \( k \) we may write \( (p \rightarrow q) \& (q \rightarrow p) \), labeling the line ‘\( j \text{ Df} \)’ and writing on its left the same numbers as are on the left of \( j \).

\[
\begin{align*}
a_1, \ldots, a_n & \quad (j) \quad (p \rightarrow q) \& (q \rightarrow p) \\
\vdots \\
a_1, \ldots, a_n & \quad (k) \quad p \leftrightarrow q \\
\text{OR} \\
a_1, \ldots, a_n & \quad (j) \quad p \rightarrow q \\
\vdots \\
a_1, \ldots, a_n & \quad (k) \quad (p \rightarrow q) \& (q \rightarrow p) \\
\end{align*}
\]
Using $\leftrightarrow$ in MacLogic

- Using the Definition strategy of MacLogic (accessed via the button), we can implement our Df. rule for $\leftrightarrow$. *Do not use $\leftrightarrow$I or $\leftrightarrow$E!*

- Using MacLogic’s Definition strategy is much simpler than using its Tautology strategy (I did that last time, which was cumbersome).

To get to Definition, first: then .

- Here is a non-trivial example: $A \leftrightarrow \sim B \vdash \sim (A \leftrightarrow B)$. Let’s try to tackle this one, using MacLogic’s Definition strategy for our Df.

- The shortest proof I’ve been able to find is 18 steps (next slide). Forbes gives a 20-stepper in his discussion of this example (p. 118).
Problem is: $A \leftrightarrow \sim B \vdash \sim (A \leftrightarrow B)$

1. $A \leftrightarrow \sim B$ \hspace{1cm} Ass
2. $A \leftrightarrow B$ \hspace{1cm} Ass
1. $(A \rightarrow \sim B) \& (\sim B \rightarrow A)$ \hspace{1cm} 1 Defn.
1. $A \rightarrow \sim B$ \hspace{1cm} 3 \&E
1. $\sim B \rightarrow A$ \hspace{1cm} 3 \&E
6. $B$ \hspace{1cm} Ass
2. $(A \rightarrow B) \& (B \rightarrow A)$ \hspace{1cm} 2 Defn.
2. $B \rightarrow A$ \hspace{1cm} 7 \&E
2,6. $A$ \hspace{1cm} 8,6 \rightarrow E
1,2,6. $\sim B$ \hspace{1cm} 4,9 \rightarrow E
1,2,6. $\Lambda$ \hspace{1cm} 10,6 \sim E
1,2. $\sim B$ \hspace{1cm} 6,11 \sim I
1,2. $A$ \hspace{1cm} 5,12 \rightarrow E
1,2. $\sim B$ \hspace{1cm} 4,13 \rightarrow E
2. $A \rightarrow B$ \hspace{1cm} 7 \&E
1,2. $B$ \hspace{1cm} 15,13 \rightarrow E
1,2. $\Lambda$ \hspace{1cm} 14,16 \sim E
1. $\sim (A \leftrightarrow B)$ \hspace{1cm} 2,17 \sim I
Sequent and Theorem Introduction: I

- You may have noticed that certain important sequents or theorems tend to get proven over and over again in different problems.

- For instance, the sequent $X \lor Y, \neg X \vdash Y$ is a very useful thing to know, as are the sequents $X \rightarrow Y, \neg Y \vdash \neg X, \land \vdash X$, and many others.

- It would be nice if we had a rule that allowed us to say “OK, I’ve proven this sequent already, so I don’t have to prove it again here”.

- We have two such rules. They are called *Sequent Introduction* (SI) for sequents, and *Theorem Introduction* (TI) for theorems.

- SI and TI allow us to avoid having to re-solve certain sub-problems that we already know how to solve. This makes proofs shorter.

- We will have a fixed list of sequents and theorems that we’ll be allowed to use in conjunction with SI and TI.
Sequent and Theorem Introduction: II

- Forbes lists a bunch of sequents and Theorems on page 123 that we may use with SI or TI. There’s a MacLogic file containing all of them.
- Here are a few of the sequents and theorems that tend to be useful:

\[
\begin{align*}
p \lor q, \neg p \vdash q; \text{ or; } p \lor q, \neg q \vdash p & \quad \text{(DS)} \\
p \rightarrow q, \neg q \vdash \neg p & \quad \text{(MT)} \\
p \vdash q \rightarrow p; \text{ or; } \neg p \vdash p \rightarrow q & \quad \text{(PMI)} \\
\vdash p \lor \neg p & \quad \text{(LEM)} \\
\neg (p \land q) \vdash \neg p \lor \neg q & \quad \text{(DEM)} \\
\neg (p \lor q) \vdash \neg p \land \neg q & \quad \text{(DEM)} \\
\neg (\neg p \lor \neg q) \vdash p \land q & \quad \text{(DEM)} \\
\neg (\neg p \land \neg q) \vdash p \lor q & \quad \text{(DEM)} \\
\land \vdash p & \quad \text{(EFQ)} \\
p \land (q \lor r) \vdash (p \land q) \lor (p \land r) & \quad \text{(DIST)}
\end{align*}
\]
Sequent and Theorem Introduction: III

• Remember the proof for #9 above: \(\vdash (A \rightarrow B) \lor (B \rightarrow A)\).

1. (1) \(\neg ((A \rightarrow B) \lor (B \rightarrow A))\)  Assumption (\(\neg I\))
2. (2) \(B\)  Assumption (\(\rightarrow I\))
3. (3) \(\neg A\)  Assumption (\(\neg I\))
4. (4) \(A\)  Assumption (\(\rightarrow I\))

2. (5) \((A \rightarrow B) \lor (B \rightarrow A)\)  4,2 \(\rightarrow I\)
2. (6) \((A \rightarrow B) \lor (B \rightarrow A)\)  5 \(\lor I\)

1,2. (7) \((A \rightarrow B) \lor (B \rightarrow A)\)  1,6 \(\neg E\)
1,2. (8) \(\neg \neg A\)  3,7 \(\neg I\)
1,2. (9) \(A\)  8 \(\text{DN}\)

1. (10) \(B \rightarrow A\)  2,9 \(\rightarrow I\)
1. (11) \((A \rightarrow B) \lor (B \rightarrow A)\)  10 \(\lor I\)
1. (12) \((A \rightarrow B) \lor (B \rightarrow A)\)  1,11 \(\neg E\)
(13) \(\neg \neg ((A \rightarrow B) \lor (B \rightarrow A))\)  1,12 \(\neg I\)
(14) \((A \rightarrow B) \lor (B \rightarrow A)\)  13 \(\text{DN}\)
Sequent and Theorem Introduction: IV

- Using TI and SI, we can obtain the following much simpler proof:

  1. \( A \lor \neg A \)  
     \hspace{1cm} \text{TI (LEM)}
  2. \( A \)  
     \hspace{1cm} \text{Assumption (\( \lor \)E)}
  2. \( B \rightarrow A \)  
     \hspace{1cm} 2 \text{ SI (PMI)}
  2. \( (A \rightarrow B) \lor (B \rightarrow A) \)  
     \hspace{1cm} 3 \text{ \( \lor \)I}
  5. \( \neg A \)  
     \hspace{1cm} \text{Assumption (\( \lor \)E)}
  5. \( A \rightarrow B \)  
     \hspace{1cm} 5 \text{ SI (PMI)}
  5. \( (A \rightarrow B) \lor (B \rightarrow A) \)  
     \hspace{1cm} 6 \text{ \( \lor \)I}
  8. \( (A \rightarrow B) \lor (B \rightarrow A) \)  
     \hspace{1cm} 1, 2, 4, 5, 7 \text{ \( \lor \)E}

- Here, LEM is the theorem \( \vdash A \lor \neg A \) (which we have already proven), and PMI stands for either of the sequents \( \neg A \vdash A \rightarrow B \) (used at line 6), or \( A \vdash B \rightarrow A \) (used at line 3), both of which we’ve proven.

- SI allows you to use (\emph{any} substitution instance of) \emph{any} sequent that you’ve already proven to make an inference at any stage of a proof.

- TI allows you to write down (\emph{any} substitution instance of) \emph{any} theorem that you have already proven at \emph{any} stage of a proof.
The Formal Definitions of SI and TI

- **Sequent Introduction** (SI). Suppose \( r_1, \ldots, r_n \vdash s \) is a substitution-instance of the sequent \( p_1, \ldots, p_n \vdash q \) which we have already proved, and that the formulae \( r_1, \ldots, r_n \) occur at lines \( j_1, \ldots, j_n \) in a proof. Then we may infer \( s \) at line \( k \), labeling the line ‘\( j_1, \ldots, j_n \) SI (Identifier)’ and writing on the left all numbers which appear on the left of lines \( j_1, \ldots, j_n \).

- **Theorem Introduction** (TI). If \( \vdash s \) is a substitution-instance of some theorem \( \vdash q \) which we have already proved, we may introduce a new line \( k \) into a proof with the formula \( s \) at it and no numbers on its left, labeling the line ‘TI (Identifier)’.

- ‘Identifier’ stands for the name of a sequent or theorem that has already been proven (e.g., MT, DS, PMI, LEM, etc). See Forbes’s list.

- Note: TI is just a special case of SI (with \( n = 0 \)).
SI and TI: A Relatively Easy Example

- Use SI/TI to find a “short” proof of: \( \sim (A \rightarrow (B \lor C)) \vdash (B \lor C) \rightarrow A \).

Problem is: \( \sim (A \rightarrow (B \lor C)) \vdash (B \lor C) \rightarrow A \)

1. \( (1) \ \sim (A \rightarrow (B \lor C)) \) 
   - Premise
2. \( (2) \ A \& \sim (B \lor C) \) 
   - \( 1 \) SI Neg-Imp1
3. \( (3) \ A \) 
   - \( 2 \) \&E
4. \( (4) \ (B \lor C) \rightarrow A \) 
   - \( 3 \) SI PMI1
SI and TI: A More Challenging Example

- Use SI/TI to find a “short” proof of: \( A \rightarrow (B \lor C) \vdash (A \rightarrow B) \lor (A \rightarrow C) \).

Problem is: \( A \rightarrow (B \lor C) \vdash (A \rightarrow B) \lor (A \rightarrow C) \)

<table>
<thead>
<tr>
<th>Step</th>
<th>Formula</th>
<th>Justification</th>
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<tbody>
<tr>
<td>1</td>
<td>( A \rightarrow (B \lor C) )</td>
<td>Premise</td>
</tr>
<tr>
<td>1</td>
<td>( \neg A \lor (B \lor C) )</td>
<td>1 SI IMP1</td>
</tr>
<tr>
<td>3</td>
<td>( \neg A )</td>
<td>Assumption (( \lor )E)</td>
</tr>
<tr>
<td>3</td>
<td>( A \rightarrow B )</td>
<td>3 SI PMI2</td>
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<tr>
<td>3</td>
<td>( (A \rightarrow B) \lor (A \rightarrow C) )</td>
<td>4 ( \lor )I_left</td>
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<tr>
<td>6</td>
<td>( B \lor C )</td>
<td>Assumption (( \lor )E)</td>
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<tr>
<td>7</td>
<td>( B )</td>
<td>Assumption (( \lor )E)</td>
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<tr>
<td>7</td>
<td>( A \rightarrow B )</td>
<td>7 SI PMI1</td>
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<tr>
<td>7</td>
<td>( (A \rightarrow B) \lor (A \rightarrow C) )</td>
<td>8 ( \lor )I_left</td>
</tr>
<tr>
<td>10</td>
<td>( C )</td>
<td>Assumption (( \lor )E)</td>
</tr>
<tr>
<td>10</td>
<td>( A \rightarrow C )</td>
<td>10 SI PMI1</td>
</tr>
<tr>
<td>10</td>
<td>( (A \rightarrow B) \lor (A \rightarrow C) )</td>
<td>11 ( \lor )I_right</td>
</tr>
<tr>
<td>6</td>
<td>( (A \rightarrow B) \lor (A \rightarrow C) )</td>
<td>6,7,9,10,12 ( \lor )E</td>
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<tr>
<td>1</td>
<td>( (A \rightarrow B) \lor (A \rightarrow C) )</td>
<td>2,3,5,6,13 ( \lor )E</td>
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</table>
The Rule of $\exists$-Introduction

Rule of $\exists$-Introduction: For any sentence $\phi \tau$, if $\phi \tau$ has been inferred at line $j$ in a proof, then at line $k$ we may infer $(\exists \nu)\phi \nu$, labeling the line ‘$j \exists I$’ and writing on its left the numbers that occur on the left of $j$.

$$a_1, \ldots, a_n \quad (j) \quad \phi \tau$$

$$\vdots$$

$$a_1, \ldots, a_n \quad (k) \quad (\exists \nu)\phi \nu \quad j \exists I$$

Where $(\exists \nu)\phi \nu$ is obtained syntactically from $\phi \tau$ by:

- Replacing one or more occurrences of $\tau$ in $\phi \tau$ by a single variable $\nu$.
- Note: the variable $\nu$ must not already occur in the expression $\phi \tau$. [This prevents double-binding, e.g., ‘$(\exists x)(\exists x)(Fx \& Gx)$’.]
- And, finally, prefixing the quantifier $(\exists \nu)$ in front of the resulting expression (which may now have both $\nu$’s and $\tau$’s occurring in it).
The Rule of $\forall$-Elimination

**Rule of $\forall$-Elimination**: For any sentence $'(\forall \nu)\phi \nu'$ and constant $\tau$, if $'(\forall \nu)\phi \nu'$ has been inferred at a line $j$, then at line $k$ we may infer $\phi \tau$, labeling the line ‘$j \forall E$’ and writing on its left the numbers that appear on the left of $j$.

$$ a_1, \ldots, a_n \quad (j) \quad (\forall \nu)\phi \nu $$

$$ \vdots $$

$$ a_1, \ldots, a_n \quad (k) \quad \phi \tau \quad j \forall E $$

Where $\phi \tau$ is obtained syntactically from $'(\forall \nu)\phi \nu'$ by:

- Deleting the quantifier prefix $'(\forall \nu)'$.
- Replacing *every occurrence* of $\nu$ in the open sentence $\phi \nu$ by *one and the same* constant $\tau$. [This prevents fallacies, e.g., $(\forall x)(Fx \to Gx) \not\models Fa \to Gb$.]
- Note: since ‘$\forall$’ means *everything*, there are *no* restrictions on which individual constant may be used in an application of $\forall E$. 

Northeastern Philosophy Natural Deduction, Cont’d 10/19/18
The Rule of \( \forall \)-Introduction: Some Background

- It is useful to think of a universal claim \( (\forall \nu)\phi \nu \) as a conjunction which asserts that the predicate expression \( \phi \) is satisfied by all objects in the domain of discourse (i.e., the conjunction \( \phi a \& (\phi b \& (\phi c \& \ldots)) \) is true).

- So, in order to be able to introduce the universal quantifier (i.e., to legitimately infer \( (\forall \nu)\phi \nu \) in a proof), we must be in a position to prove \( \phi \tau \), for any individual constant \( \tau \). This is called generalizable reasoning.

- Consider the following legitimate introduction of a universal claim:

   Problem is: \( (\forall x)(Fx \rightarrow Gx), (\forall x)Fx \vdash (\forall x)Gx \)

\[
\begin{array}{c|c|c}
1 & (1) & (\forall x)(Fx \rightarrow Gx) \\
2 & (2) & (\forall x)Fx \\
1 & (3) & Fa \rightarrow Ga \\
2 & (4) & Fa \\
1,2 & (5) & Ga \\
1,2 & (6) & (\forall x)Gx \\
\end{array}
\]

Premise
Premise
1 \ \forall E
2 \ \forall E
3,4 \ \rightarrow E
5 \ \forall I
The Rule of $\forall$-Introduction: II

- We can legitimately infer ‘$(\forall x)Gx$’ at line 6 of this proof, because our inference to ‘$Gb$’ is generalizable — i.e., we could have deduced ‘$G\tau$’, for any individual constant $\tau$ — using exactly parallel reasoning.

- However, consider the following illegitimate “$\forall$-Introduction” step:

1. $(\forall x)(Fx \to Gx)$ Premise
2. $Fb$ Premise
1. $Fb \to Gb$ 1 $\forall$E
1,2. $Gb$ 2,3 $\to$E
1,2. $(\forall x)Gx$ 4 $\forall$I NO!!

- This is not a valid inference, since $(\forall x)(Fx \to Gx), Fb \not\models (\forall x)Gx$!

- So, what went wrong? The problem is that the inference to ‘$Gb$’ at (4) is not generalizable. We can not deduce ‘$G\tau$’ — for any $\tau$ — from the premises ‘$(\forall x)(Fx \to Gx)$’ and ‘$Fb$’. We can only infer ‘$Gb$’.
The Rule of ∀-Introduction: III

Rule of ∀-Introduction: For any sentence $\phi \tau$, if $\phi \tau$ has been inferred at a line $j$, then provided that $\tau$ does not occur in any premise or assumption whose line number is on the left at line $j$, we may infer $(\forall \nu) \phi \nu$ at line $k$, labeling the line ‘$j \forall I$’ and writing on its left the same numbers as occur on the left at line $j$.

\[
\begin{align*}
  a_1, \ldots, a_n &\quad (j) \quad \phi \tau \\
  \vdots \\
  a_1, \ldots, a_n &\quad (k) \quad (\forall \nu) \phi \nu \quad j \forall I
\end{align*}
\]

Where $(\forall \nu) \phi \nu$ is obtained by:

- Replacing every occurrence of $\tau$ in $\phi \tau$ with $\nu$ and prefixing $(\forall \nu)$.
  [Again, ‘every’ prevents fallacies, e.g., $(\forall x)(Fx \rightarrow Gx) \not\equiv (\forall x)(\forall y)(Fx \rightarrow Gy)$.

- $\tau$ does not occur in any of the formulae $a_1, \ldots, a_n$. [ensures generalizability]

- $\nu$ does not occur in $\phi \tau$. [prevents double-binding]
The Rule of $\forall$-Introduction: Four Examples

Here are four examples of LMPL sequents involving the three quantifier rules we’ve learned so far ($\exists$I, $\forall$E, and $\forall$I).

1. $(\forall x)(Fx \rightarrow Gx) \vdash (\forall x)Fx \rightarrow (\forall x)Gx$
2. $\sim(\exists x)(Fx & Gx) \vdash (\forall x)(Fx \rightarrow \sim Gx)$
3. $\sim(\forall x)Fx \vdash (\exists x)\sim Fx$
4. $(\forall x)[Fx \rightarrow (\forall y)Gy] \vdash (\forall x)(\forall y)(Fx \rightarrow Gy)$
Proof of (1)

Problem is: \((\forall x)(Fx \rightarrow Gx) \vdash (\forall x)Fx \rightarrow (\forall x)Gx\)

1. \((\forall x)(Fx \rightarrow Gx)\)  Premise
2. \((\forall x)Fx\)  Assumption
1. \(Fa \rightarrow Ga\)  1 \(\forall E\)
2. \(Fa\)  2 \(\forall E\)
1,2 \(\Rightarrow Ga\)  3,4 \(\rightarrow E\)
1,2 \((\forall x)Gx\)  5 \(\forall I\)
1 \((\forall x)Fx \rightarrow (\forall x)Gx\)  2,6 \(\rightarrow I\)
Proof of (2)

Problem is: \( \neg(\exists x)(Fx\&Gx) \vdash (\forall x)(Fx\rightarrow\neg Gx) \)

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<tr>
<td>1</td>
<td>(1) ( \neg(\exists x)(Fx&amp;Gx) )</td>
<td>Premise</td>
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<td>2</td>
<td>(2) ( Fa )</td>
<td>Assumption</td>
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<td>3</td>
<td>(3) ( Ga )</td>
<td>Assumption</td>
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<td>1,5 \negE</td>
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<td>1,2</td>
<td>(7) ( \neg Ga )</td>
<td>3,6 \negI</td>
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<td>1</td>
<td>(8) ( Fa\rightarrow\neg Ga )</td>
<td>2,7 \rightarrowI</td>
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<td>(9) ( (\forall x)(Fx\rightarrow\neg Gx) )</td>
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Proof of (3)

Problem is: \( \neg (\forall x)Fx \vdash (\exists x)\neg Fx \)

<table>
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<tr>
<th>Step</th>
<th>Rule</th>
<th>Line(s)</th>
<th>Notes</th>
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<tr>
<td>1</td>
<td>(\neg(\forall x)Fx)</td>
<td>1</td>
<td>Premise</td>
</tr>
<tr>
<td>2</td>
<td>(\neg(\exists x)\neg Fx)</td>
<td>2</td>
<td>Assumption</td>
</tr>
<tr>
<td>3</td>
<td>(\neg F a)</td>
<td>3</td>
<td>Assumption</td>
</tr>
<tr>
<td>3</td>
<td>(\exists x)\neg Fx)</td>
<td>4</td>
<td>(\exists I)</td>
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<tr>
<td>2,3</td>
<td>(\neg\neg Fa)</td>
<td>5</td>
<td>2,4 (\neg E)</td>
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<td>2</td>
<td>(Fa)</td>
<td>6</td>
<td>3,5 (\neg I)</td>
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<td>2</td>
<td>(\forall x)Fx)</td>
<td>7</td>
<td>(\forall I)</td>
</tr>
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<td>1,2</td>
<td>(\neg(\exists x)\neg Fx)</td>
<td>8</td>
<td>1,8 (\neg E)</td>
</tr>
<tr>
<td>1</td>
<td>(\neg(\exists x)\neg Fx)</td>
<td>9</td>
<td>2,9 (\neg I)</td>
</tr>
<tr>
<td>1</td>
<td>(\exists x)\neg Fx)</td>
<td>10</td>
<td>10 (\neg D N)</td>
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Proof of (4)

Problem is: \((\forall x)(Fx \rightarrow (\forall y)Gy) \vdash (\forall x)(\forall y)(Fx \rightarrow Gy)\)

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<tbody>
<tr>
<td>1</td>
<td>(1) ((\forall x)(Fx \rightarrow (\forall y)Gy))</td>
<td>Premise</td>
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<td>2</td>
<td>(2) (Fa)</td>
<td>Assumption</td>
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<tr>
<td>1</td>
<td>(3) (Fa \rightarrow (\forall y)Gy)</td>
<td>1 (\forall E)</td>
</tr>
<tr>
<td>1,2</td>
<td>(4) ((\forall y)Gy)</td>
<td>3,2 (\rightarrow E)</td>
</tr>
<tr>
<td>1,2</td>
<td>(5) (Gb)</td>
<td>4 (\forall E)</td>
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<tr>
<td>1</td>
<td>(6) (Fa \rightarrow Gb)</td>
<td>2,5 (\rightarrow I)</td>
</tr>
<tr>
<td>1</td>
<td>(7) ((\forall y)(Fa \rightarrow Gy))</td>
<td>6 (\forall I)</td>
</tr>
<tr>
<td>1</td>
<td>(8) ((\forall x)(\forall y)(Fx \rightarrow Gy))</td>
<td>7 (\forall I)</td>
</tr>
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The Rule of $\exists$-Elimination: Some Background

- It is useful to think of an existential claim $\forall (\exists \nu) \phi \nu$ as a disjunction which asserts that the predicate expression $\phi$ is satisfied by at least one object in the domain (i.e., that the disjunction $\forall \phi a \lor (\phi b \lor (\phi c \lor \ldots))$ is true).

- In this way, we would expect the elimination rule for $\exists$ to be similar to the elimination rule for $\lor$. That is, we’d expect the $\exists$E rule to be similar to the $\lor$E rule. Indeed, this is the case. It’s best to start with a simple example.

- Consider the following legitimate elimination of an existential claim:

Problem is: $(\forall x)(Fx \& Gx) \vdash (\exists x)Fx$

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<thead>
<tr>
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<tbody>
<tr>
<td>1</td>
<td>(1) $(\exists x)(Fx &amp; Gx)$</td>
<td>Premise</td>
</tr>
<tr>
<td>2</td>
<td>(2) Fa &amp; Ga</td>
<td>Assumption</td>
</tr>
<tr>
<td>2</td>
<td>(3) Fa</td>
<td>2 &amp;E</td>
</tr>
<tr>
<td>2</td>
<td>(4) $(\exists x)Fx$</td>
<td>3 $\exists$I</td>
</tr>
<tr>
<td>1</td>
<td>(5) $(\exists x)Fx$</td>
<td>1,2,4 $\exists$E</td>
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</tbody>
</table>
The Rule of ∃-Elimination: II

- To derive a sentence using the ∃E rule (with some existential sentence \( (\exists \nu) \phi \nu \)) , we must first assume an instance \( \phi \tau \) of \( (\exists \nu) \phi \nu \).

- If we can deduce from this assumed instance \( \phi \tau \) — using generalizable reasoning — then we may infer outright.

- It is because our reasoning from the instance \( \phi \tau \) of \( (\exists \nu) \phi \nu \) to does not depend on our choice of constant \( \tau \) (i.e., that our reasoning from \( \phi \tau \) to is generalizable) that makes this inference valid.

- When our reasoning is generalizable in this sense, it’s as if we are showing that can be deduced from any instance \( \phi \tau \) of \( (\exists \nu) \phi \nu \).

- As such, this is just like showing that can be deduced from any disjunct of the disjunction \( \phi a \lor (\phi b \lor (\phi c \lor \ldots)) \). And, this is just like \( \lor E \) reasoning (except that ∃E only requires one assumption).
The Rule of $\exists$-Elimination: III

- Here’s an illegitimate “$\exists$-Elimination” step:

1. (1) $(\exists x)Fx$  
   Premise
2. (2) Ga  
   Premise
3. (3) Fa  
   Assumption
2, 3  (4) Fa&Ga  
   2, 3  &I
2, 3  (5) $(\exists x)(Fx&Gx)$  
   4  $\exists$I
1, 2  (6) $(\exists x)(Fx&Gx)$  
   1, 3, 5  $\exists$E  NO!!

- This is not a valid inference: $(\exists x)Fx, Ga \not\equiv (\exists x)(Fx & Gx)$!

- So, what went wrong here? The problem is that the inference to
  ‘$(\exists x)(Fx & Gx)$’ at line (5) does not use generalizable reasoning.

- We can not legitimately infer ‘$(\exists x)(Fx & Gx)$’ at line (5) from an
  arbitrary instance ‘$F\tau$’ of ‘$(\exists x)Fx$’. We must assume ‘$Fa$’ in
  particular at line (3) in order to deduce ‘$(\exists x)(Fx & Gx)$’ at line (5).
The Rule of $\exists$-Elimination: Official Definition

$\exists$-Elimination: If $\forall \nu \phi \nu$ occurs at $i$ depending on $a_1, \ldots, a_n$, an instance $\phi \tau$ of $\forall \nu \phi \nu$ is *assumed* at $j$, and is inferred at $k$ depending on $b_1, \ldots, b_u$, then at line $m$ we may infer, with label ‘$i, j, k \exists E$’ and dependencies $\{a_1, \ldots, a_n\} \cup \{b_1, \ldots, b_u\}/j$: 

\[
\begin{align*}
\quad a_1, \ldots, a_n & \quad (i) \quad (\exists \nu) \phi \nu \\
\quad \vdots & \\
\quad j \quad (j) \quad \phi \tau \quad \text{Assumption} \\
\quad \vdots & \\
\quad b_1, \ldots, b_u & \quad (k) \\
\quad \vdots & \\
\{a_1, \ldots, a_n\} \cup \{b_1, \ldots, b_u\}/j & \quad (m) \quad i, j, k \exists E
\end{align*}
\]

Provided that *all four* of the following conditions are met:

- $\tau$ (in $\phi \tau$) replaces every occurrence of $\nu$ in $\phi \nu$. [avoids fallacies]
- $\tau$ does not occur in $\forall \nu \phi \nu$. [generalizability]
- $\tau$ does not occur in. [generalizability]
- $\tau$ does not occur in any of $b_1, \ldots, b_u$, except (possibly) $\phi \tau$ itself. [generalizability]
The Rule of ∃-Elimination: Nine Examples

• Here are 9 examples of proofs involving all four quantifier rules.

1. \((∃x)\simFx ⊢ \sim(∀x)Fx\)  
2. \((∃x)(Fx → A) ⊢ (∀x)Fx → A\)  
3. \((∀x)(∀y)(Gy → Fx) ⊢ (∀x)[(∃y)Gy → Fx]\)  
4. \((∃x)[Fx → (∀y)Gy] ⊢ (∃x)(∀y)(Fx → Gy)\)  
5. \(A ∨ (∃x)Fx ⊢ (∃x)(A ∨ Fx)\)  
6. \((∃x)(Fx & ∼Fx) ⊢ (∀x)(Gx & ∼Gx)\)  
7. \((∀x)[Fx → (∀y)∼Fy] ⊢ (∃x)Fx\)  
8. \((∀x)(∃y)(Fx & Gy) ⊢ (∃y)(∀x)(Fx & Gy)\)  
9. \((∃y)(∀x)(Fx & Gy) ⊢ (∀x)(∃y)(Fx & Gy)\)
Proof of (1)

Problem is: $(\exists x)\lnot Fx \vdash \lnot (\forall x)Fx$

1. $(\exists x)\lnot Fx$  
   Premise
2. $(\forall x)Fx$  
   Assumption
3. $\lnot Fa$  
   Assumption
4. $Fa$  
   2 $\forall E$
5. $\lnot$  
   3, 4 $\lnot E$
6. $\lnot$  
   1, 3, 5 $\exists E$
7. $\lnot (\forall x)Fx$  
   2, 6 $\lnot I$
Proof of (2)

Problem is: $(\exists x)(Fx \to A) \vdash (\forall x)Fx\to A$

1. $(\exists x)(Fx \to A)$  Premise
2. $(\forall x)Fx$  Assumption
3. $Fa \to A$  Assumption
2. $(Fa) 2 \forall E$
2,3  $(A) 3,4 \rightarrow E$
1,2  $(A) 1,3,5 \exists E$
1  $(\forall x)Fx \to A) 2,6 \rightarrow I$
Proof of (3)

Problem is: \((\forall x)(\forall y)(Gy \rightarrow Fx) \vdash (\forall x)((\exists y)Gy \rightarrow Fx)\)

1 \hspace{1cm} (1) \hspace{1cm} (\forall x)(\forall y)(Gy \rightarrow Fx) \hspace{1cm} \text{Premise}

2 \hspace{1cm} (2) \hspace{1cm} (\exists y)Gy \hspace{1cm} \text{Assumption}

3 \hspace{1cm} (3) \hspace{1cm} Gy \hspace{1cm} \text{Assumption}

1 \hspace{1cm} (4) \hspace{1cm} (\forall y)(Gy \rightarrow Fa) \hspace{1cm} \text{1 \ \forall E}

1 \hspace{1cm} (5) \hspace{1cm} Gy \rightarrow Fa \hspace{1cm} \text{4 \ \forall E}

1,3 \hspace{1cm} (6) \hspace{1cm} Fa \hspace{1cm} \text{5,3 \ \rightarrow E}

1,2 \hspace{1cm} (7) \hspace{1cm} Fa \hspace{1cm} \text{2,3,6 \ \exists E}

1 \hspace{1cm} (8) \hspace{1cm} (\exists y)Gy \rightarrow Fa \hspace{1cm} \text{2,7 \ \rightarrow I}

1 \hspace{1cm} (9) \hspace{1cm} (\forall x)((\exists y)Gy \rightarrow Fx) \hspace{1cm} \text{8 \ \forall I}
Proof of (4)

Problem is: \((\exists x)(Fx \rightarrow (\forall y)Gy) \vdash (\exists x)(\forall y)(Fx \rightarrow Gy)\)

1. \((\exists x)(Fx \rightarrow (\forall y)Gy)\) Premise
2. \(Fa \rightarrow (\forall y)Gy\) Assumption
3. \(Fa\) Assumption
2,3 4. \((\forall y)Gy\) 2,3 \(\rightarrow E\)
2,3 5. \(Gb\) 4 \(\forall E\)
2 6. \(Fa \rightarrow Gb\) 3,5 \(\rightarrow I\)
2 7. \((\forall y)(Fa \rightarrow Gy)\) 6 \(\forall I\)
2 8. \((\exists x)(\forall y)(Fx \rightarrow Gy)\) 7 \(\exists I\)
1 9. \((\exists x)(\forall y)(Fx \rightarrow Gy)\) 1,2,8 \(\exists E\)
Proof of (5)

Problem is: $A \rightarrow (\exists x)Fx \vdash (\exists x)(A \lor Fx)$

1. $A \rightarrow (\exists x)Fx$  
   Premise
2. $A$  
   Assumption
2. $A \lor Fa$  
   2 $\lor I$
2. $(\exists x)(A \lor Fx)$  
   3 $\exists I$
5. $(\exists x)Fx$  
   Assumption
6. $Fa$  
   Assumption
6. $A \lor Fa$  
   6 $\lor I$
6. $(\exists x)(A \lor Fx)$  
   7 $\exists I$
5. $(\exists x)(A \lor Fx)$  
   5, 6, 8 $\exists E$
1. $(\exists x)(A \lor Fx)$  
   1, 2, 4, 5, 9 $\lor E$
Proof of (6)

Problem is: (∃x)(Fx&¬Fx) ⊨ (∀x)(Gx&¬Gx)

1. (1) (∃x)(Fx&¬Fx)  Premise
2. (2) Fa&¬Fa  Assumption
3. (3) ¬Gb  Assumption
2. (4) ¬Fa  2 &E
2. (5) Fa  2 &E
2. (6) ∨  4,5 ¬E
2. (7) ¬¬Gb  3,6 ¬I
2. (8) Gb  7 DN
9. (9) Gb  Assumption
2. (10) ¬Gb  9,6 ¬I
2. (11) Gb&¬Gb  8,10 &I
2. (12) (∀x)(Gx&¬Gx)  11 ∀I
1. (13) (∀x)(Gx&¬Gx)  1,2,12 ∃E
Proof of (7)

Problem is: \((\forall x)(Fx \rightarrow (\forall y) \sim Fy) \vdash \sim (\exists x)Fx\)

1 \hfill (1) \((\forall x)(Fx \rightarrow (\forall y) \sim Fy)\) \hspace{1cm} \text{Premise}
2 \hfill (2) \((\exists x)Fx\) \hspace{1cm} \text{Assumption}
3 \hfill (3) Fa \hspace{1cm} \text{Assumption}
1,3 \hfill (4) Fa \rightarrow (\forall y) \sim Fy \hspace{1cm} 1 \ \forall E
1,3 \hfill (5) (\forall y) \sim Fy \hspace{1cm} 4,3 \ \rightarrow E
1,3 \hfill (6) \sim Fa \hspace{1cm} 5 \ \forall E
1,3 \hfill (7) \Lambda \hspace{1cm} 6,3 \ \sim E
1,2 \hfill (8) \Lambda \hspace{1cm} 2,3,7 \ \exists E
1 \hfill (9) \sim (\exists x)Fx \hspace{1cm} 2,8 \ \sim I
Proof of (8)

Problem is: $(\forall x)(\exists y)(Fx \& Gy) \vdash (\exists y)(\forall x)(Fx \& Gy)$

1. $(\forall x)(\exists y)(Fx \& Gy)$  \hspace{1cm} \text{Premise}
2. $(\exists y)(Fa \& Gy)$  \hspace{1cm} 1 $\forall$E
3. $Fa \&Gb$  \hspace{1cm} \text{Assumption}
4. $(\exists y)(Fc \& Gy)$  \hspace{1cm} 1 $\forall$E
5. $Fc\&Gd$  \hspace{1cm} \text{Assumption}
6. $Fc$  \hspace{1cm} 5 $\&$E
7. $Fc$  \hspace{1cm} 4,5,6 $\exists$E
8. $Gb$  \hspace{1cm} 3 $\&$E
9. $Fc\&Gb$  \hspace{1cm} 7,8 $\&$I
10. $(\forall x)(Fx \&Gb)$  \hspace{1cm} 9 $\forall$I
11. $(\exists y)(\forall x)(Fx \& Gy)$  \hspace{1cm} 10 $\exists$I
12. $(\exists y)(\forall x)(Fx \& Gy)$  \hspace{1cm} 2,3,11 $\exists$E
Proof of (9)

Problem is: $(\exists y)(\forall x)(Fx & Gy) \vdash (\forall x)(\exists y)(Fx & Gy)$

1. $(\exists y)(\forall x)(Fx & Gy)$ Premise
2. $(\forall x)(Fx & Gb)$ Assumption
3. $Fa & Gb$ 2 $\forall$E
4. $(\exists y)(Fa & Gy)$ 3 $\exists$I
5. $(\exists y)(Fa & Gy)$ 1, 2, 4 $\exists$E
6. $(\forall x)(\exists y)(Fx & Gy)$ 5 $\forall$I