

Working with LMPL Interpretations

Philosophy 4515
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1 Working with *Given* LMPL Interpretations

Consider the following (*given*) LMPL interpretation \mathcal{I}_1 :

	F	G	H	I	J	
(\mathcal{I}_1)	α	+	+	-	+	-
	β	-	-	-	+	+
	γ	+	-	-	-	+

In other words, the interpretation \mathcal{I}_1 has the following features: $\mathcal{D} = \{\alpha, \beta, \gamma\}$, $\text{Ext}(F) = \{\alpha, \gamma\}$, $\text{Ext}(G) = \{\alpha\}$, $\text{Ext}(H) = \emptyset$ (where, \emptyset is *the null set*), $\text{Ext}(I) = \{\alpha, \beta\}$, and $\text{Ext}(J) = \{\beta, \gamma\}$.

Question: What are the \mathcal{I}_1 -truth-values of ①–⑥?

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|--|---|
| ① $\sim Ja$ | ④ $(\forall x)[Jx \rightarrow (Gx \vee Fx)]$ |
| ② $Fc \rightarrow Ic$ | ⑤ $(\exists x)Gx \rightarrow (\forall y)(Fy \vee Gy)$ |
| ③ $(\exists x)(Jx \leftrightarrow Hx)$ | ⑥ $(\exists y)(\forall x)[Gy \ \& \ (Jx \rightarrow (Ix \vee Fx))]$ |

Solutions:

- ① ' $\sim Ja$ ' is *true* on \mathcal{I}_1 . This is because ' Ja ' is *false* on \mathcal{I}_1 , since $\alpha \notin \text{Ext}(J)$.
- ② ' $Fc \rightarrow Ic$ ' is *false* on \mathcal{I}_1 . This is because its *antecedent* ' Fc ' is *true* on \mathcal{I}_1 , since $\gamma \in \text{Ext}(F)$; but its *consequent* ' Ic ' is *false* on \mathcal{I}_1 , since $\gamma \notin \text{Ext}(I)$.
- ③ ' $(\exists x)(Jx \leftrightarrow Hx)$ ' is *true* on \mathcal{I}_1 . The *instances* of ' $(\exists x)(Jx \leftrightarrow Hx)$ ' on \mathcal{I}_1 are: (i) ' $Ja \leftrightarrow Ha$ ', (ii) ' $Jb \leftrightarrow Hb$ ', and (iii) ' $Jc \leftrightarrow Hc$ '. Instances (ii) and (iii) are *false* on \mathcal{I}_1 (*why?*). But, instance (i) is *true* on \mathcal{I}_1 , because ' Ja ' and ' Ha ' are *both false* on \mathcal{I}_1 , since $\alpha \notin \text{Ext}(J)$ and $\alpha \notin \text{Ext}(H)$.¹
- ④ ' $(\forall x)[Jx \rightarrow (Gx \vee Fx)]$ ' is *false* on \mathcal{I}_1 . The *instances* of ' $(\forall x)[Jx \rightarrow (Gx \vee Fx)]$ ' on \mathcal{I}_1 are as follows: (i) ' $Ja \rightarrow (Ga \vee Fa)$ ', (ii) ' $Jb \rightarrow (Gb \vee Fb)$ ', and (iii) ' $Jc \rightarrow (Gc \vee Fc)$ '. Instances (i) and (iii) are *true* on \mathcal{I}_1 (*why?*). But, instance (ii) is *false* on \mathcal{I}_1 . This is because ' Jb ' is *true* on \mathcal{I}_1 , since $\beta \in \text{Ext}(J)$; but ' $Gb \vee Fb$ ' is *false* on \mathcal{I}_1 , since $\beta \notin \text{Ext}(G)$ and $\beta \notin \text{Ext}(F)$.²
- ⑤ ' $(\exists x)Gx \rightarrow (\forall y)(Fy \vee Gy)$ ' is \perp on \mathcal{I}_1 . The *antecedent* ' $(\exists x)Gx$ ' of this conditional is \top on \mathcal{I}_1 , because its *instance* ' Ga ' is \top on \mathcal{I}_1 , since $\alpha \in \text{Ext}(G)$. But, the *consequent* ' $(\forall y)(Fy \vee Gy)$ ' of this conditional is *false* on \mathcal{I}_1 , because its *instance* ' $Fb \vee Gb$ ' is *false* on \mathcal{I}_1 , since $\beta \notin \text{Ext}(G)$ and $\beta \notin \text{Ext}(F)$.
- ⑥ ' $(\exists y)(\forall x)[Gy \ \& \ (Jx \rightarrow (Ix \vee Fx))]$ ' is *true* on \mathcal{I}_1 . The three *instances* of ⑥ on \mathcal{I}_1 are as follows:
 - (i) ' $(\forall x)[Ga \ \& \ (Jx \rightarrow (Ix \vee Fx))]$ '. This instance of ⑥ is \top on \mathcal{I}_1 . The *instances* of (i) are as follows: (i.1) ' $Ga \ \& \ (Ja \rightarrow (Ia \vee Fa))$ ', (i.2) ' $Ga \ \& \ (Jb \rightarrow (Ib \vee Fb))$ ', and (i.3) ' $Ga \ \& \ (Jc \rightarrow (Ic \vee Fc))$ '. (i.1) is \top on \mathcal{I}_1 since *both* ' Ga ' [$\alpha \in \text{Ext}(G)$], and ' $Ja \rightarrow (Ia \vee Fa)$ ' [$\alpha \notin \text{Ext}(J)$] are \top on \mathcal{I}_1 . (i.2) is \top on \mathcal{I}_1 since *both* ' Ga ' and ' $Jb \rightarrow (Ib \vee Fb)$ ' [$\beta \in \text{Ext}(J)$ and $\beta \in \text{Ext}(I)$] are \top on \mathcal{I}_1 . (i.3) is \top on \mathcal{I}_1 since *both* ' Ga ' and ' $Jc \rightarrow (Ic \vee Fc)$ ' [$\gamma \in \text{Ext}(J)$ and $\gamma \in \text{Ext}(F)$] are \top on \mathcal{I}_1 .
 - (ii) ' $(\forall x)[Gb \ \& \ (Jx \rightarrow (Ix \vee Fx))]$ '. This instance of ⑥ is \perp on \mathcal{I}_1 , because ' Gb ' is *false* on \mathcal{I}_1 , since $\beta \notin \text{Ext}(G)$. So, *none* of the three instances of ' $(\forall x)[Gb \ \& \ (Jx \rightarrow (Ix \vee Fx))]$ ' is *true* on \mathcal{I}_1 (*why?*).
 - (iii) ' $(\forall x)[Gc \ \& \ (Jx \rightarrow (Ix \vee Fx))]$ '. This instance of ⑥ is \perp on \mathcal{I}_1 , because ' Gc ' is *false* on \mathcal{I}_1 , since $\gamma \notin \text{Ext}(G)$. So, *none* of the three instances of ' $(\forall x)[Gc \ \& \ (Jx \rightarrow (Ix \vee Fx))]$ ' is *true* on \mathcal{I}_1 (*why?*).

All instances of (i) are \top on \mathcal{I}_1 . \therefore (i) is \top on \mathcal{I}_1 . \therefore One instance of ⑥ is *true* on \mathcal{I}_1 . \therefore ⑥ is \top on \mathcal{I}_1 .

¹Remember, it only takes *one true instance* (on \mathcal{I}) of the *existential* claim ' $(\exists v)\phi v$ ' to make ' $(\exists v)\phi v$ ' *true* on \mathcal{I} .

²Remember, it only takes *one false instance* (on \mathcal{I}) of the *universal* claim ' $(\forall v)\phi v$ ' to make ' $(\forall v)\phi v$ ' *false* on \mathcal{I} .

2 Constructing LMPL Interpretations to Prove \neq Claims

Problem #1. Show that:

$$(1) \quad (\forall x)(Fx \rightarrow Gx), (\forall x)(Fx \rightarrow Hx) \neq (\forall x)(Gx \rightarrow Hx).$$

Solution. In order to prove (1), we need to construct an interpretation \mathcal{I} on which ‘ $(\forall x)(Fx \rightarrow Gx)$ ’ and ‘ $(\forall x)(Fx \rightarrow Hx)$ ’ are both true, but ‘ $(\forall x)(Gx \rightarrow Hx)$ ’ is false. We proceed in several steps.

- **Step 1:** We begin — *provisionally* — with the smallest possible domain $\mathcal{D} = \{\alpha\}$.
- **Step 2:** We make sure that the object α is a *counterexample* to the conclusion ‘ $(\forall x)(Gx \rightarrow Hx)$ ’. That is, we make sure that the *instance* ‘ $G\alpha \rightarrow H\alpha$ ’ of the conclusion is *false* on \mathcal{I} . So, we must have $\alpha \in \text{Ext}(G)$, but $\alpha \notin \text{Ext}(H)$. We can achieve this by making $\text{Ext}(G) = \{\alpha\}$, and $\text{Ext}(H) = \emptyset$.
- **Step 3:** At the same time, we try to make *both* of the premises ‘ $(\forall x)(Fx \rightarrow Gx)$ ’ and ‘ $(\forall x)(Fx \rightarrow Hx)$ ’ *true* on \mathcal{I} . In this case, we can make both premises true simply by ensuring that $\alpha \notin \text{Ext}(F)$. The simplest way to do this is to stipulate that $\text{Ext}(F) = \emptyset$ — which yields the following interpretation:

$$(I_2) \quad \begin{array}{c|ccc} & F & G & H \\ \hline \alpha & - & + & - \end{array}$$

We have discovered an interpretation \mathcal{I}_2 on which ‘ $(\forall x)(Fx \rightarrow Gx)$ ’ and ‘ $(\forall x)(Fx \rightarrow Hx)$ ’ are both true, but ‘ $(\forall x)(Gx \rightarrow Hx)$ ’ is false (*demonstrate this!*). Therefore, claim (1) is true.³

Problem #2. Show that:

$$(2) \quad (\exists x)(Fx \& Gx), (\exists x)(Fx \& Hx), (\forall x)(Gx \rightarrow \sim Hx) \neq (\forall x)[Fx \leftrightarrow (Gx \vee Hx)].$$

Solution. In order to prove (2), we need to construct an interpretation \mathcal{I} on which ‘ $(\exists x)(Fx \& Gx)$ ’, ‘ $(\exists x)(Fx \& Hx)$ ’, and ‘ $(\forall x)(Gx \rightarrow \sim Hx)$ ’ are all true, but ‘ $(\forall x)[Fx \leftrightarrow (Gx \vee Hx)]$ ’ is false.

- **Step 1:** We begin — *provisionally* — with the smallest possible domain $\mathcal{D} = \{\alpha\}$.
- **Step 2:** We make sure that the object α is a *counterexample* to the conclusion ‘ $(\forall x)[Fx \leftrightarrow (Gx \vee Hx)]$ ’. So, we make its *instance* ‘ $F\alpha \leftrightarrow (G\alpha \vee H\alpha)$ ’ *false* on \mathcal{I} . There are several ways to do this. One way: $\alpha \in \text{Ext}(F)$, $\alpha \notin \text{Ext}(G)$, and $\alpha \notin \text{Ext}(H)$. So far, we have $\text{Ext}(F) = \{\alpha\}$, and $\text{Ext}(G) = \text{Ext}(H) = \emptyset$.
- **Step 3:** Now, we must try to make *all three* of the premises (i) ‘ $(\exists x)(Fx \& Gx)$ ’, (ii) ‘ $(\exists x)(Fx \& Hx)$ ’, and (iii) ‘ $(\forall x)(Gx \rightarrow \sim Hx)$ ’ *true* on \mathcal{I} . In order to make (i) true on \mathcal{I} , we must ensure that there is some object in the domain \mathcal{D} which satisfies *both* predicates ‘ F ’ and ‘ G ’. But, since α must *not* satisfy both ‘ F ’ and ‘ G ’, this means we will need to *add another object* β to our domain \mathcal{D} , such that: $\beta \in \text{Ext}(F)$, and $\beta \in \text{Ext}(G)$. Now, we have $\text{Ext}(F) = \{\alpha, \beta\}$, $\text{Ext}(G) = \{\beta\}$, and $\text{Ext}(H) = \emptyset$. All that remains is to ensure that premises (ii) and (iii) are also true on \mathcal{I} . In order to make (ii) true on \mathcal{I} , we’ll need to make sure that there is some object in \mathcal{D} which satisfies *both* predicates ‘ F ’ and ‘ H ’. We could *try* to make β satisfy *all three* predicates ‘ F ’, ‘ G ’, and ‘ H ’. But, if we were to do this, then premise (iii) would become *false* on \mathcal{I} , since its *instance* ‘ $G\beta \rightarrow \sim H\beta$ ’ would then be false on \mathcal{I} . Thus, we’ll need to *add a third object* γ to \mathcal{D} such that: $\gamma \in \text{Ext}(F)$, $\gamma \notin \text{Ext}(G)$, and $\gamma \in \text{Ext}(H)$ — success:

$$(I_3) \quad \begin{array}{c|ccc} & F & G & H \\ \hline \alpha & + & - & - \\ \beta & + & + & - \\ \gamma & + & - & + \end{array}$$

We have discovered an interpretation \mathcal{I}_3 on which ‘ $(\exists x)(Fx \& Gx)$ ’, ‘ $(\exists x)(Fx \& Hx)$ ’, and ‘ $(\forall x)(Gx \rightarrow \sim Hx)$ ’ are all true, but ‘ $(\forall x)[Fx \leftrightarrow (Gx \vee Hx)]$ ’ is false (*demonstrate this!*). Therefore, claim (2) is true.

³When you’re asked to prove a claim like (1), you must do *two* things: (i) *Report* an interpretation (like \mathcal{I}_2) which serves as a counterexample to the validity of the LMPL argument-form, and (ii) *Demonstrate* that your interpretation *really* is a counterexample — *i.e.*, show that your interpretation makes all the premises true and the conclusion false, using the methods on the front page of this handout. You do **not** need to explain the process which led to the *discovery* of the interpretation.