It is customary to setup probability calculi in such a way that:

- Logical/Semantic notions are *presupposed*.
- Conditional Probability is *defined in terms of* unconditional probability (*via* a Ratio Formula).

Suppose we have a function $\Pr : \mathcal{L} \mapsto \mathbb{R}$ — where $\mathcal{L}$ is a sentential language having connectives $\wedge, \vee, \neg$ — satisfying the following four constraints, for all statements p, q of $\mathcal{L}$.

(1) $0 \leq \Pr(p) \leq 1$
(2) If p and q are *equivalent*, then $\Pr(p) = \Pr(q)$.
(3) $\Pr(\neg p) = 1 - \Pr(p)$
(4) $\Pr(p \vee q) = \Pr(p) + \Pr(q)$, if p and q are *incompatible*.

We then *define* conditional probability $\Pr(\cdot \mid \cdot)$ in terms of unconditional probability $\Pr(\cdot)$, *via* the following Ratio Formula.

(5) $$\Pr(p \mid q) \overset{\text{def}}{=} \frac{\Pr(p \wedge q)}{\Pr(q)} \text{ } \textit{if} \Pr(q) > 0.$$

Interestingly, (1)–(5) can be adapted to handle both classical, bivalent logic/semantics and trivalent logic/semantics where statements p of $\mathcal{L}$ can be true, false, or neither [2, 4].

The first trick is to add the following clarification, where $\mathrm{Tr}(p)$ asserts that p is true, and $\mathrm{TV}(p)$ asserts that p has a truth-value.

(6) $$\Pr(p) := \frac{\Pr(\mathrm{Tr}(p))}{\Pr(\mathrm{TV}(p))}, \textit{ if } \Pr(\mathrm{TV}(p)) > 0.$$

The second trick is to define *equivalence* and *incompatibility* in the following way (in both bivalent and trivalent semantics).

- p and q are *equivalent* if every assignment that makes p true (false) makes q true (false) and *vice versa*.
- p and q are *incompatible* if there is no assignment where both p and q are true.

One reason to favor a trivalent approach is that it can accommodate the addition of an Adams-style conditional — without being subject to Lewisian triviality results [10, 4].

To be more precise, suppose we add a conditional connective ($\to$) to our language $\mathcal{L}$, and we want it to obey the following strong form of Adams's Thesis, for all p, q, x in $\mathcal{L}$.

Adams. $\Pr(p \to q \mid x) = \Pr(q \mid p \wedge x)$

Adding **Adams** to classical probability calculus yields triviality [7]. *Not so* on the trivalent approach. Indeed, on the trivalent approach, **Adams** is a *theorem*, for bivalent p, q, x [10, 4].

This is because we have the following *modified version* of the Ratio formula for trivalent conditional probability.

(7) $$\Pr(z \mid x) \overset{\text{def}}{=} \frac{\Pr(\mathrm{Tr}(z \wedge x))}{\Pr(\mathrm{TV}(z \wedge x) \wedge \mathrm{Tr}(x))} \textit{ if } \Pr(\mathrm{TV}(z \wedge x) \wedge \mathrm{Tr}(x)) > 0.$$

Specifically, if $z = p \to q$, then — for bivalent p, q, x — we have

Adams (trivalent). For all bivalent p, q, x in $\mathcal{L}$,

$$\Pr(p \to q | x) = \frac{\Pr(\mathrm{Tr}((p \to q) \wedge x))}{\Pr(\mathrm{TV}((p \to q) \wedge x) \wedge \mathrm{Tr}(x))} = \frac{\Pr(p \wedge q \wedge x)}{\Pr(p \wedge x)}.$$

Suppose, instead of (1)–(4), we adopt the following four axioms.

(I) $\Pr(p) = 0$ if $p \vdash \bot$.
(II) $\Pr(p) = 1$ if $\top \vdash p$.
(III) If $p \vdash q$, then $\Pr(p) \leq \Pr(q)$.
(IV) $\Pr(p) + \Pr(q) = \Pr(p \vee q) + \Pr(p \wedge q)$.

Here, we allow "$\vdash$" to be interpreted either in terms of *classical* logical entailment $\vdash_C$ or *intuitionistic* logical entailment $\vdash_I$.[1]

In this way — together with (6) our ratio def. of $\Pr(\cdot \mid \cdot)$ — we can obtain either classical or intuitionistic probability calculus [21].[2]

Like classical PC, Intuitionistic PC is in tension with **Adams**. Adding **Adams** to intuitionistic probability calculus yields

No Disconfirmation. For all e, h, k in $\mathcal{L}$,

$$\Pr(h \mid k \wedge e) \geq \Pr(h \mid k).$$

[1] *Paraconsistent* probability can also be explicated in the (I)–(IV) style [1].
[2] Some authors in the "Traditional Approach" camp (most notably [22]) take *updating* — rather than *supposing* — to be fundamental for $\Pr(\cdot \mid \cdot)$.

In fact, we only need the following three general principles about conditional probability (+ associativity and commutativity of $\wedge$) in order to derive **No Disconfirmation** from **Adams** [16].

(C1) $0 \le \Pr(p \mid q) \le 1$
(C2) $\Pr(p \mid p \wedge q) = 1$
(C3) $\Pr(p \wedge q \mid r) = \Pr(p \mid q \wedge r) \cdot \Pr(q \mid r)$

So, in order to avoid **No Disconfirmation**, we must reject at least one of {(C1), (C2), (C3), **Adams**}. (C1) is simply a choice of finite scale for conditional probability. So, rejecting (C1) seems absurd.

Rejecting (C2) also seems absurd. This would be tantamount to (for some statements p and q) supposing $p \wedge q$ and — under that very supposition — *accepting* "*might not p*." [24, 25]

It would seem that the only viable option for rejecting **No Disconfirmation** while accepting **Adams** is to reject (C3).

This is what the trivalent approach [10, 4], and two other recent approaches to semantics & probability of indicatives do [3, 8].

Popper [18, 19, 12] proposes an alternative approach, which

- Does *not* presuppose Logical/Semantic notions.
- Does *not* define Conditional Probability in terms of unconditional probability.

In the classical case, Popper [19] gives the following six axioms for conditional probability — as a function Pr: $\mathcal{L} \times \mathcal{L} \mapsto \mathbb{R}$.

Nontrivial. $(\exists p)(\exists q)\,[\Pr(p \mid q) \neq \Pr(p \mid p)]$

Identity. $\Pr(p \mid p) \le \Pr(q \mid q)$

Monotony. $\Pr(p \wedge q \mid r) \le \Pr(p \mid r)$

Product [(C3)]. $\Pr(p \wedge q \mid r) = \Pr(p \mid q \wedge r) \cdot \Pr(q \mid r)$

Sum. $\Pr(p \mid r) + \Pr(q \mid r) = \Pr(p \wedge q \mid r) + \Pr(p \vee q \mid r)$

Negation. $(\exists p)\,[\Pr(p \mid r) \neq \Pr(r \mid r)] \Longrightarrow [\Pr(q \mid r) + \Pr(\neg q \mid r) = \Pr(r \mid r)]$

Popper calls this system $\mathbb{B}$. All the standard laws of classical conditional probability [*e.g.*, (C1)–(C3)] follow from $\mathbb{B}$ [18, 19] — *without assuming anything about the logic/semantics of $\mathcal{L}$.*

Popper defines the following relation between statements of $\mathcal{L}$.

$$p \preceq q \stackrel{\text{def}}{=} (\forall r)\,[\Pr(p \mid r) \le \Pr(q \mid r)]$$
$$p \simeq q \stackrel{\text{def}}{=} (p \preceq q) \,\&\, (q \preceq p)$$

$\mathbb{B}$ implies that $\simeq$ is an equivalence relation, which (a) *induces a Boolean algebra* $\mathscr{B}$ on $\mathcal{L}$ (*viz.*, the one determined by $\dashv\vDash$), and (b) is such that $p \simeq q$ ensures *full inter-substitutivity* of p and q.

In this sense, we can see classical logic/semantics as *emerging* from constraints on conditional probability [6].

If we add **Adams** to $\mathbb{B}$, then $p \to q \simeq p \supset q$, and $\Pr(\cdot \mid \cdot)$ is 2-valued [11, 20], *i.e.*, for all p, q in $\mathcal{L}$, $\Pr(p \mid q) = 0$ or $\Pr(p \mid q) = 1$.

Next, I will examine several weakenings of Popper's classical theory $\mathbb{B}$. These can be used to formulate autonomous probability calculi for various subsystems of classical logic.[3]

[3] In addition to logics *weaker* than classical logic, the Popperian Approach can also be applied to *extensions/strenghenings* of classical logic [14].

The system $\mathbb{M}$ is given by the following six axioms.

Nonzero. $(\exists p)(\exists q)\,[\Pr(p \mid q) \neq 0]$

Downbound. $0 \le \Pr(p \mid q)$

Upbound. $\Pr(p \mid q) \le \Pr(r \mid r)$

Monotony. $\Pr(p \wedge q \mid r) \le \Pr(p \mid r)$

Product. $\Pr(p \wedge q \mid r) = \Pr(p \mid q \wedge r) \cdot \Pr(q \mid r)$

$\mathbb{M}$ implies that $\simeq$ is an equivalence relation, which (a) *induces a semilattice* structure $\mathscr{M}$ on $\mathcal{L}$ (with $\wedge$ as meet), and (b) is such that $p \simeq q$ ensures *full inter-substitutivity* of p and q.[4]

$\mathbb{M}$ + **Adams** $\Rightarrow$ **No Disconfirmation** — see Extras slides.

$\mathbb{M}$ + **Sum** = $\mathbb{D}$, where $\simeq$ induces a *distributive lattice* $\mathscr{D}$ on $\mathcal{L}$.[5]

[4] In subsystems of $\mathbb{B}$, we also need $p \simeq q \Longrightarrow (\forall r)\,[\Pr(r \mid p) = \Pr(r \mid q)]$, so as to ensure $p \simeq q$ guarantees inter-substitutivity of p and q on the right-hand sides of $\Pr(\cdot \mid \cdot)$'s — see page 10 of [19] for discussion.

[5] Narens [17] takes a Traditional Approach to lattice-theoretic probability.

If we add the following two axioms to $\mathbb{D}$, then $\simeq$ induces a *distributive lattice with zero/bottom element* ($\bot$) $\mathscr{D}_\bot$ on $\mathcal{L}$. I will call the resulting probability calculus $\mathbb{D}_\bot$.

Zero$_1$. $\Pr(p \mid q) \leq \Pr(q \mid \bot)$

Zero$_2$. $[\Pr(p \mid q) \neq \Pr(p \mid p)] \Longrightarrow [\Pr(p \mid \bot) + \Pr(\bot \mid q) = \Pr(p \mid p)]$

In order to get from $\mathbb{D}_\bot$ to intuitionistic probability calculus $\mathbb{H}$ (with intuitionistic conditional $\dashrightarrow$), we need two more axioms.

Conditional$_1$. $\Pr(p \dashrightarrow q \mid p \wedge r) = \Pr(q \mid p \wedge r)$

Conditional$_2$. $(p \wedge q) \preceq r \Longrightarrow p \preceq (q \dashrightarrow r)$

Then, the algebra induced by $\simeq$ on $\mathcal{L}$ is a Heyting Algebra $\mathscr{H}$ [12]. And, if we add **Negation** to $\mathbb{H}$, this brings us back to $\mathbb{B}$.

The axiomatization of intuitionistic probability calculus remains controversial. Some authors assume **Adams** rather than **Conditional**$_1$ [15], which leads to **No Disconfirmation** [16, 21].

In [16, 15], $\Pr(h \mid k) \leq \Pr(h \mid e \wedge k)$ is interpreted as implying only "if we can *prove* h from k, we can *prove* h from $e \wedge k$."

On this interpretation, **No Disconfirmation** seems benign. But, as Weatherson [21] points out, a *Bayesian* with "intuitionsitic tendencies" may reasonably reject **No Disconfirmation.**

They may interpret $\Pr(h \mid k) \leq \Pr(h \mid e \wedge k)$ as implying that *no e can disconfirm any h, relative to any background corpus k.*

On either the Traditional or the Popperian approach, we need to be clear on how we are understanding $\Pr(p \mid q)$. I follow Weatherson in taking a Bayesian stance here.

That is, I prefer to understand $\Pr(p \mid q)$ as measuring a *(rational) degree of belief in p, on the (indicative) supposition that q* [5].

The "semantic/logical" interpretation of $\Pr(\cdot \mid \cdot)$ is less clear to me. Some authors seem to interpret $\Pr(p \mid q)$ as "the degree to which q is deducible from p" [11]. And, some have an externalist epistemic ("evidential probability") interpretation [23].

I will close with three questions that I'm hoping to answer.

- Can the Popperian Approach be used for the trivalent case? That is, can we write down axioms for $\Pr(\cdot \mid \cdot)$ which
  (a) give us the properties of $\Pr(\cdot \mid \cdot)$ we seek (*e.g.*, **Adams** without triviality or **No Disconfirmation**), *and*
  (b) induce (*via* $\simeq$) the semantical structures of, *e.g.*, [2, 4].
- Can the Popperian Approach be applied to other non-classical systems that have been developed recently — especially in connection with indicative conditionals and epistemic modals. For instance, the systems being explored by Goldstein & Santorio [8] and Holliday & Mandelkern [9]?
- More generally, I like the idea of taking *conditional probability judgments regarding pairs of sentences* (of $\mathcal{L}$) as fundamental [3]. It would be nice if we could write down axioms for *that*, and let *those* determine the underlying logic/semantics for $\mathcal{L}$ — *à la* Popper. But, *can this be done?*

Here is a proof (sketch) of **No Disconfirmation** in Popper's system $\mathbb{M}$. We will first prove (C1), (C2), and a **Lemma.**

(C1) $0 \leq \Pr(p \mid q) \leq 1$

Proof.

We just need to prove $\Pr(p \mid q) \leq 1$. In light of **Upbound**, it suffices to show that $\Pr(p \mid p) = 1$. By **Upbound**, $\Pr(p \mid p)$ is independent of p. So, set $\Pr(p \mid p) = k$ for all p. Then, from **Downbound** and **Monotony**, we have

$$\Pr(p \mid p \wedge p) \leq k = \Pr(p \wedge p \mid p \wedge p) \leq \Pr(p \mid p \wedge p)$$

$$\Pr(p \mid p \wedge (p \wedge p)) \leq k = \Pr(p \wedge (p \wedge p) \mid p \wedge (p \wedge p)) \leq \Pr(p \mid p \wedge (p \wedge p))$$

$$\therefore \Pr(p \mid p \wedge p) = \Pr(p \mid p \wedge (p \wedge p)) = k$$

Then, by **Product**, we have

$$k = \Pr(p \wedge p \mid p \wedge p) = \Pr(p \mid p \wedge (p \wedge p)) \cdot \Pr(p \mid p \wedge p) = k^2$$

Hence, either $k = 0$ or $k = 1$. By **Product**, **Upbound**, and **Monotony** (×2)

$$k \cdot \Pr(x \mid z) = \Pr(x \wedge z \mid x \wedge z) \cdot \Pr(x \mid z) = \Pr((x \wedge z) \wedge x \mid z) \leq \Pr(x \wedge z \mid z) \leq \Pr(x \mid z) \leq k$$

Thus, if $k = 0$, then $\Pr(x \mid z) = 0$ for all x, z, which contradicts **Nonzero**. □

(C2) $\Pr(x \mid x \wedge z) = 1 = \Pr(x \mid z \wedge x)$

Proof.

Since by (C1) the value of $\Pr(x \mid z \wedge y)$ lies in $[0, 1]$, we obtain the following inequality immediately from **Product**

(i) $$\Pr(x \wedge z \mid y) \leq \Pr(z \mid y)$$

By $\Pr(p \mid p) = 1$, **Monotony**, and **Upbound**, we have

(ii) $$1 = \Pr(x \wedge z \mid x \wedge z) \leq \Pr(x \mid x \wedge z) \leq 1$$

and likewise, using (i) in place of **Monotony**, we have

(iii) $$1 = \Pr(z \wedge x \mid z \wedge x) \leq \Pr(x \mid z \wedge x) \leq 1$$

Together, (ii) and (iii) imply (C2). □

Lemma. If $\Pr(x \mid y \wedge z) = 1$, then $\Pr(y \mid z) \leq \Pr(x \mid z)$.

Proof.

By **Product**, we have

(iv) $$\Pr(x \mid y \wedge z) \cdot \Pr(y \mid z) = \Pr(x \wedge y \mid z)$$

Suppose $\Pr(x \mid y \wedge z) = 1$.

From (iv) and **Monotony**, we may deduce

$$\Pr(y \mid z) = \Pr(x \wedge y \mid z) \leq \Pr(x \mid z)$$

Therefore, $\Pr(y \mid z) \leq \Pr(x \mid z)$. □

No Disconfirmation. $\Pr(h \mid k \wedge e) \geq \Pr(h \mid k)$

Proof.

(v) $\Pr(h \mid e \wedge (h \wedge k)) = 1$

- by (C2) + associativity & commutativity of $\wedge$.

(vi) $\Pr(e \to h \mid h \wedge k) = 1$

- by (v) + **Adams**.

(vii) $\Pr(h \mid k) \leq \Pr(e \to h \mid k)$

- by (vi) + **Lemma** (where $x := e \to h$, $y := h$, $z := k$).

(viii) $\Pr(h \mid k) \leq \Pr(h \mid k \wedge e)$

- by (vii) + **Adams** + commutativity of $\wedge$.

Popper and Miller [19] give a formal proof of the associativity and commutativity of $\wedge$ from the axioms of $\mathbb{M}$ (+ the assumption that $p \simeq q \Longrightarrow (\forall r)\,[\Pr(r \mid p) = \Pr(r \mid q)]$ — see footnote 4). □

[1] J. Bueno-Soler and W. Carnielli, *Paraconsistent Probabilities: . . .*, 2016.
[2] J. Cantwell, *The Laws Of Non-Bivalent Probability*, 2006.
[3] I. Ciardelli and A. Ommundsen, *Probabilities of conditionals: updating Adams*, 2022.
[4] P. Egré, L. Rossi, J. Sprenger, *Certain and Uncertain Inference with Trivalent Conditionals*, 2022.
[5] B. Eva, T. Shear and B. Fitelson, *Four Approaches to Supposition*, 2021.
[6] H. Field, *Logic, Meaning, and Conceptual Role*, 1977.
[7] B. Fitelson, *The Strongest Possible Lewisian Triviality Result*, 2016.
[8] S. Goldstein and P. Santorio, *Probability for Epistemic Modalities*, 2021.
[9] W. Holliday and M. Mandelkern, *The Orthologic of Epistemic Modals*, 2022.
[10] D. Lassiter, *What We Can Learn From How Trivalent Conditionals Avoid Triviality*, 2020.
[11] D. Miller, *Reconditioning the Conditional*, 2016.
[12] D. Miller, *How Does Probability Theory Generalize Logic?*, 2004.
[13] D. Miller and K. Popper, *Deductive Dependence*, 1986.
[14] C. Morgan, *There is a Probabilistic Semantics for Every Extension of Classical Sentence Logic*, 1981.
[15] C. Morgan and H. Leblanc, *Probabilistic Semantics for Intuitionistic Logic*, 1983.
[16] C. Morgan and E. Mares, *Conditionals, Probability and Non-Triviality*, 1995.
[17] L. Narens, *Probabilistic Lattices*, 2014.
[18] K. Popper, *The Logic of Scientific Discovery*, 1959.
[19] K. Popper and D. Miller, *Contributions to the Formal Theory of Probability*, 1994.
[20] P. Roeper and H. Leblanc, *Probability Theory and Probability Logic*, 1999.
[21] B. Weatherson, *From Classical to Intuitionistic Probability*, 2003.
[22] R. Williams, *Probability and nonclassical logic*, 2016.
[23] T. Williamson, *Knowledge and its Limits*, 2002.
[24] S. Yalcin, *Epistemic Modals*, 2007.
[25] S. Yalcin, *Bayesian Expressivism*, 2012.