SEMANTICAL ANALYSIS OF MODAL LOGIC II.
NON-NORMAL MODAL PROPOSITIONAL CALCULI

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This paper continues the investigations of Kripke [63]. Not only is Kripke [63] presupposed; the reader is advised to have it on hand for ready reference. The notations and terminology of that paper are used freely; in particular, P, Q, R, ... are atomic formulae, and A, B, C, ... are arbitrary formulae built from them using the connectives &,  ˘ ,  ˘ .

All propositional calculi in this paper have the same formation rules as those of Kripke [63]. However, they in general will lack the rule of necessitation of Kripke [63] and thus will not be normal. Hence (in our opinion), they are intuitively somewhat unnatural; but nevertheless they have an elegant model theory. Among these systems, notably, Lewis's S2 and S3; this paper extends the results of Kripke [63] to these and other systems. The results of this paper were announced in Kripke [63, abstract].

All systems considered here will be formulated with axiom schemata; substitution is a derived rule.

1. Generalities; Halldén's property. By R1, or modus ponens, we mean the rule: A, A  ˘ B  ˘ B.

Theorem 1. Let PL be any propositional calculus containing the theorem schema A  ˘ (A  ˘ ... A) and its converse (A  ˘ ... A)  ˘ A. If modus ponens is an admissible (derivable) rule of PL, PL can be recursively axiomatized with modus ponens as sole rule of inference.

Proof. (Essentially given in Craig [58].) Form PL' as follows:

Axioms: (A  ˘ ... A)  ˘ A (any number of conjuncts A), and B  ˘ ... B (m conjuncts), where m is the Gödel number of a proof of B in PL. R1 is the only rule. Then clearly PL contains PL', and PL' contains PL because if B is any theorem of PL and m is the Gödel number of a proof thereof, B  ˘ ... B and B  ˘ ... B  ˘ B (m conjuncts) are both provable in PL', whence B is by R1. Q.E.D.

S2, S3, and most systems considered below have an "incompleteness" property of Halldén [51]: some formula A ˘ B is derivable, where A and B have no propositional variables in common and neither A nor B is derivable. Halldén essentially showed that no system with his property can possess a normal characteristic matrix.

Theorem 2. Let PL be a propositional calculus containing all classical tautologies and with R1 as sole postulated rule. Let A and B be formulae with no propositional variables in common. Suppose A ˘ B is provable, and let PL' (PL") be formed by adding all substitution instances of A (of B) as a new axiom schema to PL. Then PL=PL' ˘ PL".

Proof. Let C be a theorem of PL' and of PL". We need to derive it in PL. Since ˘ C in PL', there are finitely many instances A1, ..., Am of A from which C is derivable in PL'. Hence, by the deduction theorem, ˘ A1 ˘ ... ˘ Am ˘ C in PL. Similarly, ˘ B1 ˘ ... ˘ Bn ˘ C in PL for certain substitution instances B1, ..., Bn of B. So we need only derive A1 ˘ ... ˘ Am ˘ B1 ˘ ... ˘ Bn in PL. By relettering if necessary, we can assume that A1, ..., Am, as well as A, contain no variables in common with B; no matter how B is thus relettered, we can still derive A ˘ B in PL. Since A and B have no propositional variables in common, by substitution, we get ˘ A1 ˘ B in PL (i=l, ..., m); hence ˘ A1 ˘ ... ˘ Am ˘ B. Since in this latter formula the disjuncts still have no variable in common, substituting again we get ˘ (A1 ˘ ... ˘ Am) v Bj (j=l, ..., n), and hence ˘ (A1 ˘ ... ˘ Am) v (B1 ˘ ... ˘ Bn). Q.E.D.

Theorem 3. If PL is a propositional calculus, it possesses a normal characteristic matrix if and only if the following four conditions hold:

(a) PL is consistent (i.e., ˘ A ˘ ~ A).
(b) R1 is an admissible rule of PL.
(c) PL contains all classical tautologies.
(d) PL does not possess Halldén's property.

The necessity of the conditions is clear. The proof of sufficiency, which we omit, proceeds by constructing a normal characteristic matrix by Lindenbaum's method.

Although we are assuming that all propositional calculi have the formation rules of Kripke [63], it is clear that the theorems above hold for a wider class of formation rules. Further, Theorem 2 applies, e.g., to systems containing the Hilbert-Bernays positive logic (in place of classical logic).
2. Propositional calculi considered. All systems considered contain all tautologies of classical propositional logic as axioms; these axioms will not be listed explicitly.

The system E2 of Lemmon [57] is characterized by the axioms:

A1. $\square A \supset A$,
A2. $\square (A \supset B) \supset (\square A \supset \square B)$,
and the rules R1 and

(Eb) If $\vdash A \supset B$, then $\vdash \square A \supset \square B$.

We get a system E3 if we replace A3 in E2 by the stronger axiom:

(1) $\square (A \supset B) \supset (\square A \supset \square B)$.

It is easy to show that (Eb) is equivalent to $A \supset B \supset A \supset \square B$.

Now by Theorem 1, E2 and E3 can be reformulated with R1 as sole rule of inference. Alternatively, we can get more natural axiomatizations of these two systems by dropping the rule (Eb), and replacing it by the clause: if $B$ is an axiom and $A$ is any formula, $\square A \supset \square B$ is an axiom. Then, with R1 as sole postulated rule, we can obtain $B / \square A \supset \square B$ as a derived rule, and hence (Eb) (by the inference: $A \supset B / \square A \supset (A \supset B)$, hence $\square A \supset \square B$ by A3 and truth-functional logic).

Given a formulation of E2 (E3) with R1 as sole (primitive) rule, define S2 (S3) as the result of adding to this formulation the schema $\square (A \supset B)$. Further, for a formulation of E2 with R1 as sole rule, define $S2^a$ by adding thereto:

(2) $\square^n (A \supset B)$ ($0 < n < \infty$).

1 This proviso was inadvertently neglected in Kripke [63]. In connection with Kripke [63], Footnote 2 on p. 69, we should mention that an outline of Hintikka's work has since appeared as Hintikka [61]; and that the "pupils" of T. J. Smiley mentioned in that footnote can now be more accurately specified as Francis Drake.

2 We can prove that if $B$ is derivable, so is $\square A \supset \square B$, by induction on the length of a proof of $B$ in the axiomatized version of E2 (E3). If $B$ is an axiom, the desired result is clear. If $B$ is derived from $C$ and $C \supset B$ by R1, we can assume by the inductive hypothesis that $\square A \supset \square C$ and $\square A \supset (\square C \supset \square B)$ are derivable. By axiom A3 and truth-functional logic, we obtain $\vdash \square A \supset (\square C \supset \square B)$ from the second of these; and from this together with the first we get the desired $\vdash \square A \supset \square B$.

Note that the rule $B / \square A \supset \square B$ is a trivial consequence of (Eb) (since $A \supset B$ is derivable from B). So the axiomatized version of E2 (E3) is clearly contained in E2 (E3). Above we indicated how to show that it contains E2 (E3).

Here $\square^n$ indicates a sequence of $n$ necessity signs. Thus $S2^0$ is E2; $S2^1$ is E2.

Now if $A$ is a theorem of E2 (E3), so is $A \supset A \cdot \cdot \cdot A$, hence by (Eb) $\vdash (\square A \cdot A \cdot \cdot A) \supset A$. Hence $\vdash A$ in S2 (S3). Similarly (using $n$ applications of (Eb)), $\square^n (A \supset A) \supset A$ is a theorem of E2, so by R1 $\square^n A$ is a theorem of $S2^n$. In Theorem 6 below we will show conversely that if $\vdash \square^n A$ in $S2^n$, then $\vdash A$ in E2.

If we define $S2^a$ by adding (2) for all $n$ as schemata to a formulation of E2 with R1 as sole rule, it is easily shown that if $B$ is an axiom of $S2^a$, $\vdash B$ in $S2^a$ (for every axiom is either $\square^n (A \supset A)$ or a theorem of E2). Hence by induction on the length of a proof of $B$ in $S2^a$, $\vdash B$ in $S2^a$ implies $\vdash B$; and thus $S2^a$ is equivalent to Feys-von Wright's M (cf. Kripke [63]).

Notice that in E3, since $\vdash \square B \supset (A \supset B)$, by two applications of (1) to the consequent, we get $\vdash \square B \supset (\square A \supset \square B)$, and hence $\vdash \square A \supset \square B \supset \square A \supset (\square A \supset B)$. Hence it can be shown that, if we defined $S3^a$ by adding (1) to $S2^a$, $S3^a$ would collapse into S4. (Some intuitive motivation for the facts that $S3^2 = S4$, $S2^a = M$ will become apparent from the model theory below.)

The systems $S2^a$, E3, and S3 will be discussed, model-theoretically, below. In the rest of this section, we mention other systems, and their relations to the previous systems and to each other. The Falsum system is the system with R1 as sole rule and the following single axiom:

(3) $\sim \square A$.

Axiom (3) reduces "$\square$" to a falsum operator. It can easily be verified that E2 and E3 admit an interpretation with "$\square$" as the falsum operator. So does Lemmon's E4, whose rules are R1 and (Eb) and whose axioms are A1, A3, and

A4. $\square A \supset \square A$.

Lemmon's E5 collapses into S5; the present writer has proposed instead a system E6 defined by A1, A3,

A2. $\square A \supset \square B \supset \square \sim \square B$.

3 We have just handled the case where $B$ is an axiom. If $B$ is derived by R1 from $C$ and $C \supset B$, we can assume by inductive hypothesis that $\square C$ and $\square (C \supset B)$ are derivable. From the second of these we get by A3 $\vdash \square C \supset \square B$, hence using $\square C$ and R1, we obtain the desired $\vdash \square B$. 
and R1 and (Eb). This system is still contained in the Falsum system.

Now since \( \vdash B \cdot A \supset A \), by (Eb) \( \vdash B \supset (A \supset A) \) in E2; i.e., \( \vdash \square (A \supset A) \) \( \sim \) \( \square B \). Hence, applying Theorem 2 to E2 (E3) reformulated with R1 as sole rule, E2 (E3) is the intersection of S2 (S3) with the Falsum system. Theorem 2 further implies that E4 (E5) is analogously related to S4 (S5).

We remark that all systems so far considered (except the Falsum) are contained in the Trivial system, which is based on the axioms \( \square A \supset A \) and \( A \supset \square A \). R1 is the only rule.

The Łukasiewicz L-modal system is axiomatized by the axioms A1 and
\[(4) \quad A \supset B \cdot \circ \square A \supset \square B.\]

R1 is the only rule (cf. Łukasiewicz [53]). Curry’s Gentzen rules for possibility in Curry [52] turn out to be equivalent to the L-modal system, although by the time Curry [52] was published, he no longer advocated this system as natural. Another formulation of the system L-modal system contains all the E-systems.

3. Models. Any attempt to develop a model theory for E2 and E3 must face the problem that no formula of the form \( \square B \) is provable in E2 or E3, not even \( \square (A \supset A) \). If, as in Kripke [63], however, we were to define a formula \( \square B \) as true in a “world” \( H, B \) iff \( B \) is true in every \( H, \square B \) “possible relative to” \( H, B \), and evaluated the truth-functions by truth-tables, then since \( A \supset A \) would clearly be true in any world \( H, \square B \) would have to be true in any world \( H, B \). Similar difficulties arise in S2 and S3 (in S2s), owing to the fact that \( \square \square (A \supset A) \square (\square \square (A \supset A)) \) is unprovable.

We observe, however, that \( \vdash \square B \supset \square (A \supset A) \) in E2, and hence \( \vdash \square A \supset A \supset \square B \); if \( A \supset A \) is not necessary, then nothing is. This leads us to divide the “possible worlds” into two classes; the “normal worlds”, in which necessity is evaluated according to the Leibnizian prescription of Kripke [63]; and the non-normal worlds, in which \( \square B \) is always false.

We define, then, a model structure (m.s.) to be an ordered triple \((G, K, R)\), where \( K \) is a set, \( G \in K \), and \( R \) is a relation defined on \( K \) with the following quasi-reflexivity property: If \( H, R, H, H \in K \), then \( H, R, H, R \). An element of \( K \) (a “world”) \( H \) such that \( H, R, H \) is called normal.

Alternatively, we could define a model structure as a quadruple \((G, K, R, N)\), where \( K \) is a set, \( G \in K \), \( N \subseteq K \), and \( R \) is a relation on \( K \) which is reflexive on \( N \). The normal worlds are the elements of \( N \). This definition leads to a slightly more general class of models at the expense of one more primitive; its advantages will become apparent in Section 7.

A m.s. is also called an E2 m.s. if \( R \) is transitive, a m.s. \((G, K, R)\) is an E3 m.s. An E2 (E3) m.s. in which \( G \) is normal is called an S2 (S3) m.s.

Temporarily let \( Q \) be the following relation between elements of \( K \) and non-negative integers: \( H Q n \) iff either \( n = 0 \) and \( H = G \) or \( n > 1 \) and there exist \( H_0, \ldots, H_n \in K \) such that \( H_0 = G, H_n = H, \) and \( H_i R H_{i+1} \) \((0 < i < n)\). Define the depth of an element \( H \) of \( K \) to be the least \( n \) such that \( H Q n \); if no such \( n \) exists, we say the depth is infinite.

We define an E2 m.s. to be an S2s m.s. iff every \( H \in K \) of depth \( < n \) is normal. For \( n = 0 \) \((n - 1)\), this coincides with the previous notion of E2 (S2) m.s.

An E2 (E3, S2s, S3) model \( \varphi \) for a wff \( A \) of E2 (E3, S2s, S3) is a binary function \( \varphi(P, H) \) associated with a given E2 (E3, S2s, S3) m.s. \((G, K, R)\). As in Kripke [63], the first variable \( P \) ranges over atomic subformulas of \( A \), while the second variable \( H \) ranges over elements of \( K \). The range of \( \varphi \) is the set \([T, F]\).

Given a model \( \varphi \) associated with a m.s. \((G, K, R)\), we define for any subformula \( B \) of \( A \) a value \( \varphi(B, H) \). If \( B \) is atomic, the value has already been given. If \( \varphi(B, H) \) and \( \varphi(C, H) \) have been defined, for each \( H \), then define \( \varphi(B \land C, H), \varphi(B, H), \varphi(B, H) \) as in Kripke [63]. We define \( \varphi(\square B, H) = T \) if \( H \) is normal and \( \varphi(B, H^1) = T \) for all \( H^1 \) such that \( H R H \); otherwise, \( \varphi(\square B, H) = F \). (Hence, in particular, \( \varphi(\square B, H) = F \) automatically, if \( H \) is not normal.)

As in Kripke [63], a formula \( A \) is true in a model \( \varphi \) associated with a m.s. \((G, K, R)\), \( \varphi(A, G) = T \). \( A \) is valid (for S2s, E3, S3) iff it is true in every \((S2s, E3, S3)\) model. Notice that here, unlike the situation in Kripke [63], the distinguished element \( G \) in a m.s. \((G, K, R)\) plays an essential role in the definitions. For example, in an S2 model \( \varphi \) on a m.s. \((G, K, R)\), \( \varphi(\square(\square A)), G \) must equal \( T \), since \( G \) is normal in an S2 m.s.; hence \( \square(\square A) \) is valid in S2.

We can define connected models and tree models as in Kripke [63]. As in Kripke [63], we could confine ourselves to connected models...
without essentially changing the theory. Notice that, in a connected m.s. \((G, K, R)\), every \(H \in K\) has a finite depth; and that, in a connected m.s. where \(R\) is transitive, every \(H \in K\) has depth \(< 1\). Hence, if we define a m.s. \((G, K, R)\) to be an \(S2^\infty\) m.s. if every \(H \in K\) of finite depth is normal, then a connected \(S2^\infty\) m.s. is an M m.s. (cf. Kripke [63]).

If we define a \(S3^\infty\) m.s. as an \(E3\) m.s. where every \(H \in K\) of depth \(< 2\) is normal, then a connected \(S3^\infty\) m.s. is an \(S4\) m.s. Conversely, under these definitions, an M \((S4)\) m.s. is an \(S2^\infty (S3^\infty)\) m.s. These facts provide the model-theoretic motivation for the results in Section 2 that \(S2^\infty = M, S3^\infty = S4\).

4. Semantic tableaux. We can adopt almost all of the notions of Kripke [63] for tableau constructions without change. Again, we have, at a given stage of the construction, a system of alternative sets of tableaux. Each alternative set \(S\) is ordered by a relation \(R\) which now need only be quasi-reflexive. The ordered set \(S\) can, in a natural way, be given the form of a tree; the origin \(t_0\) of the tree is the (descendant in \(S\)) of the main tableau. Replacing \(G\) by \(t_0\), we can define the depth \(n\) of a tableau \(t\) in \(S\), just as it was previously defined for a world: the depth \(n\) of \(t\) is the least \(n\) such that either \(n = 0\) and \(t = t_0\), or \(n > 1\) and there exist \(t_1, \ldots, t_n\) such that \(t = t_i, R t_{i+1} (0 < i < n)\).

Now a tableau \(t\) of an alternative set \(S\) is called normal iff \(t R t\). We require \(R\) to be quasi-reflexive: \(t R t_2\) implies \(t R t_1\). In \(E3\) and \(S3\) constructions, we further require that \(R\) be transitive. In \(S2\) and \(S3\) constructions, the main tableau \(t_0\) must be normal. In \(S2^\infty\) constructions any tableau of depth \(< n\) is normal. Further, for all systems we require that a tableau with a formula \(\square B\) on the left be normal (since, correspondingly, \(\varphi(\square B, H) = T\) implies that \(H\) is normal). In short, as in Kripke [63], the properties assumed for \(R\) parallel those of \(R\); and, as in Kripke [63], the relation \(R\) is to hold in a set only if the preceding requires it to hold.

All rules except \(Yr\) can be adopted without change\(^4\) from Kripke [63]. \(Yr\) now reads:

**Yr.** If \(\square A\) appears on the right of a normal tableau \(t\), we start out a new tableau \(t'\), with \(A\) on the right, and such that \(t R t'\).

\(^4\) For \(Ar\), the italicized condition in Kripke [63], p. 73, included there to ensure reflexivity of \(R\), should be dropped in favor of: "Further if \(t R t\), we stipulate that \(t R t\)."

Notice that \(Yr\), as stated, actually renders the "requirement" of quasi-reflexivity superfluous. For if \(t R t\) is not required for some reason other than quasi-reflexivity (e.g., occurrence of some formula \(\square B\) on the left, or being of depth \(< n\) in an \(S2^\infty\) construction), then \(Yr\) will forever be inapplicable to \(t\), so that no tableau \(t'\) with \(t R t'\) will ever be introduced, and thus quasi-reflexivity will hold vacuously. Hence, in an \(E2\) or \(E3\) (an \(S3\), an \(S2^\infty\)) construction, a tableau \(t\) will be normal iff it has a formula \(\square B\) on the left (either has some formula \(\square B\) on the left or is the main tableau, either has some formula \(\square B\) on the left or is of depth \(< n\)).

**Theorem 4.** \(A\) is valid in \(E2\) (\(E3, S3, S2^\infty\)) iff the construction for \(A\) is closed (in the appropriate system).

The proof of this theorem is exactly like that of the corresponding theorem in Kripke [63], Section 3.2.

4.1. Trees and a reformulation of the rules. As in Kripke [63], each of the alternative sets of a stage of a construction can be given the form of a tree. In fact, if, following Kripke [63], we define \(t R t_2\) to hold iff \(t_1, t_2\) are descendants in \(S\) of tableaux \(t_1'\) and \(t_2'\), such that \(t_2'\) was introduced by an application of \(Yr\) to \(t_1'\), then \(S\) is a tree relation on \(S\). Following Kripke [63], Section 3.3., we raise the question whether the rules can be reformulated in terms of \(S\), in such a way that a rule applied to a tableau \(t\) can affect at most \(t\) and tableaux contiguous to \(t\) (Kripke [63], p. 81). Notice that, if we replace \(R\) by \(S\), we can define the depth of a tableau \(t\) in terms of \(S\), rather than \(R\); for \(S2^\infty\) constructions this notion of depth will coincide with the original one (but for \(E3\) and \(S3\), where \(R\) is transitive, it will not). Now, in terms of \(S\), we can define a tableau \(t\) of an \(S2^\infty\) construction to be normal iff it contains a formula \(\square A\) on the left or is of depth \(< n\), for an \(E3\) construction iff it contains a formula \(\square A\) on the left, and for an \(S3\) construction it contains a formula \(\square A\) on the left or is the main tableau of its alternative set. We can then reformulate all rules except \(Y1\) simply by replacing \(R\) by \(S\) (for \(Ar\) we drop the condition intended to ensure quasi-reflexivity; cf. Kripke [63], p. 80). For \(S2^\infty\), \(Y1\) now becomes like \(Y1\) for \(M\) in Kripke [63], p. 81:

**Y1.** Let \(\square A\) appear on the left of \(t\). Then put \(A\) on the left of \(t\) and of every tableau \(t'\) such that \(t S t'\).
For S3 and E3, we need to reformulate Y1, so as to obtain a surrogate for the transitivity property of R. We base our reformulation on the following fact, easily verified for a model φ on an E3 m.s. (G, K, R):

If \( \phi(\Box A, H_1) \rightarrow T \), \( H_1 \in K \), \( H_2 \) is a normal element of \( K \), and \( H_1 R H_2 \), then \( \phi(\Box A, H_2) \rightarrow T \). So Y1 becomes:

**Y1.** Let \( \Box A \) appear on the left of a tableau \( t \). Then: (1) put \( A \) on the left of \( t \); (2) put \( A \) on the left of every tableau \( t' \) such that \( t \leq t' \); (3) put \( \Box A \) on the left of every normal tableau \( t' \) such that \( t \leq t' \).

Notice that if \( \Box A \) appears on the left of \( t \) and \( t \leq t' \), then \( t' \) must be normal (otherwise it could not be a descendant of a tableau to which \( Y_r \) was applied, so that \( t \leq t' \) would be impossible). Hence by Y1, we put \( \Box A \) on the left of \( t' \), and thus we put \( A \) on the left of \( t' \).

This process can be iterated an arbitrary number of times, so we get the effect of transitivity (cf. the analogous discussion for S4, Kripke [63], p. 81).

The equivalence of the "S-formulation" to the "R-formulation" can be proved without much trouble. The "S-formulation" will be used throughout Section 5.

**5. Completeness theorem.**

**5.1. Consistency property.** We prove the completeness theorem for all systems S2\(^n \) (\( 0 < n < \infty \)) and for E3 and S3. We first need to show that every provable formula of a given system is valid (in the appropriate axiom schemata directly, and then verify the rules R1 and (Eb). R1 is trivial.

To verify (Eb), suppose we are given an E3 (E3) m.s. (G, K, R), and an associated model \( \phi \) such that \( \phi(\Box A \supset \Box B, G) = F \), so that \( \Box A \supset \Box B \) is not valid in E3 (E3): we need to verify that \( A \supset B \) is not valid in E3 (E3). Since \( \phi(\Box A \supset \Box B, G) = F \), \( \phi(\Box A, G) = T \), and hence G is normal. Hence, since \( \phi(\Box B, G) = F \), there exists \( H_0 \in K \) such that \( \\phi(B, H_0) = F \) and \( G R H_0 \). Since \( G \supset H_0 \), and \( \phi(\Box A, G) = T \), \( \phi(A, H_0) = T \). So \( \phi(A \supset B, H_0) = F \). Now \( (H_0, K, R) \) is an E2 (E3) m.s., and we define a model \( \phi'(P, H) = \phi(P, H) \) for all \( H \in K \). That is, \( \phi \) and \( \phi' \) are the same binary functions, but one is associated with \( (G, K, R) \) and the other with \( (H_0, K, R) \). Hence, since \( \phi(A \supset B, H_0) = F \), \( \phi'(A \supset B, H_0) = F \), and \( \phi' \) falsifies \( A \supset B \). Q.E.D. So we have now verified that all theorems of E2 (S3) are valid in E3 (E3).

In S2\(^n \), all axioms are theorems of E2 (hence valid in E2 and a fortiori in S2\(^n \)) or are of the form \( \Box^n(A \supset A) \). The latter are easily verified in S2\(^n \) to be valid on account of the restriction (in an S2\(^n \) m.s. (G, K, R)) that every \( H \in K \) of depth < \( n \) be normal. The rule R1, which is the only rule of S2\(^n \), is verified trivially. Similarly, for S3, \( \Box (A \supset A) \) is obviously valid, since G must be normal in an S3 m.s. (G, K, R), and the other verifications are trivial.

5.2. Completeness property. As in Kripke [63], we prove that if the construction based on S for A is closed, A is valid. The rank of a tableau, the associated and the characteristic formula, are as in Kripke [63]. As in Kripke [63], we need to prove the following:

**Lemma.** If \( A_0 \) is the characteristic formula of the initial stage of an S2\(^n \) (E3 or S3) construction, and \( B_0 \) is the characteristic formula of any stage, then \( \vdash A_0 \supset B_0 \) in E2 (E3).

**Proof.** As in Kripke [63], we prove that the characteristic formula of any stage \( A_m \) implies the characteristic formula \( A_{m+1} \) of the next stage. In fact, we prove that, in any S2\(^n \) construction, \( \vdash A_m \supset A_{m+1} \) is already provable in E2; and similarly that, for an E3 or S3 construction, \( \vdash A_m \supset A_{m+1} \) is already provable in E3. We can repeat all the "preliminary observations" in the proof of the corresponding lemma in Kripke [63], noting that the inference from \( \vdash X \supset Y \) to \( \vdash \Box X \supset \Box Y \) (Kripke [63], p. 84, lines 13-16), which was justified in Kripke [63] by an appeal to the rule R2, is justified in E2 and E3 by (Eb). And in fact, all the cases considered in Kripke [63] except Y1 can be repeated here without change. (It is notable that the revision of Yr makes no difference to the verification of the case \( \vdash \Box X \rightarrow \Box Y \) in fact, the case would go through even if the applicability of Yr were not restricted to normal tableaux.) So we verify only Y1:

**Case Y1.** In S2\(^n \), we have \( \Box A \) on the left of a tableau \( t \), and we are required to put \( A \) on the left of \( t \) and of every tableau \( t' \) such that \( t \leq t' \). As in Kripke [63], putting \( A \) on the left of \( t \) itself is simply justified by the axiom \( \Box A \supset A \). To justify putting \( A \) on the left of \( t' \), where \( t \leq t' \), observe (as in Kripke [63]) that if the characteristic formula of \( t \) is \( X \supset \Box A \supset B \), where \( B \) is the characteristic formula of \( t' \), putting \( A \) on the left of \( t' \) changes this characteristic formula of \( t \) to \( X \supset \Box A \supset (B \supset A) \). We can justify this transformation (as in Kripke [63]) if we can prove \( \Box A \supset (B \supset \Box A) \) in E2. Well, we certainly
A in on A. reducts to the desired of a set is stipulated to be normal even if it lacks such a formula; but only if it has some formula 0. But by the axiom A3, restated in terms of possibility, we need to verify further that it is legitimate to put 0.

But by (Eb), we can justify putting 0.

Now let to be the main tableau of the alternative set t. That is to say, there are tableaux \( t_0, \ldots, t_m \) such that each alternative set at this stage contains a closed tableau. Since the characteristic formula Xo of \( \tau \) has the form \( B \land \lnot B \), where B occurs on both the left and the right of \( \tau \), so clearly \( \vdash \sim X_p \). Assuming \( \vdash \sim X_t \), we have \( \vdash \sim X_t \) and \( \vdash \sim X_t \). Since \( \vdash \sim X_t \), the tableaux \( t_{i-1} \) must be normal; for otherwise the rule \( Y_r \) would never have been applied to it, contrary to the fact that \( t_{i-1} \). So (remembering that we are dealing with E2 or E3), \( t_{i-1} \) has some formula \( \Box B \) on the left. Hence its characteristic formula \( X_{i-1} \) has the form \( Y \land \Box B \land \sim X_t \). But since \( \vdash \sim X_t \), a fortiori \( \vdash \Box B \land \sim X_t \), hence by (Eb), \( \vdash \Box B \land \sim X_t \), and hence \( \vdash \sim (\Box B \land \sim X_t) \). This implies \( \vdash \sim X_{i-1} \), as required.

So we have proved \( \vdash \sim D_t \) for each i. Hence we obtain \( \vdash \sim (D_1 \lor \ldots \lor D_m) \).

This proof worked for E2 and E3. What about \( S_2^a \) and \( S_3^a \)? We treat \( S_2, S_3, \) and \( S_2^a; \) the treatment of \( S_2^a \) for arbitrary \( a \) is similar. In \( S_2 \) and \( S_3 \), we can no longer assert that a normal tableau must have some formula \( \Box B \) on the left; the main tableau \( t_0 \) is an exception. But the characteristic axiom \( \Box (C \land C) \) of \( S_2 \) and \( S_3 \) shows that, for any formula \( Y \), \( \vdash Y \land \Box (C \land C) \land Y \). By contraposition, \( \vdash \sim (\Box (C \land C) \land Y) \). Now we can infer \( \vdash \sim X_0 \) in \( S_2 \) and \( S_3 \). Thus if \( Y \) is \( X_0 \), the preceding proof certainly shows that \( \vdash \sim (\Box (C \land C) \land X_0) \). In \( S_2 \) and \( S_3 \), we can no longer be sure that \( t_0 \) and \( t_1 \) contain a formula \( \Box B \) on the left, even if both are normal, although we can be sure of this for any tableau of depth \( \geq 2 \). But the characteristic formula \( X_0 \) of \( t_0 \) has the form \( Y \land \Box X_1 \). Using the axiom \( \Box (C \land C) \) of \( S_2^a \), we get \( \vdash Y \land \Box X_1 \land \Box (C \land C) \land \Box (C \land C) \lor X_1 \). (If \( p=0 \), \( X_1 \) drops out, and we simply use \( \vdash Y \land \Box X_1 \).) Now we can infer (by the preceding reasoning), successively in \( E_2 \), \( \sim (\Box (C \land C) \land X_1) \), and then \( \vdash \sim (Y \land \Box (C \land C) \land \Box (C \land C) \land X_1) \). Combining this with the previous result, we get \( \vdash \sim (Y \land X_1) \), i.e., \( \vdash \sim X_0 \). Similarly for \( S_2^a \); and then the reasoning goes through as before.
is an E2 countermodel \( \varphi \) on a m.s. \((G', K', R')\) for it. Define a m.s. \((G', K', R')\), where \( K' = K \cup \{G'\} \), \( G' \notin K' \), and, for \( H' \) and \( H'_a \in K' \), \( H' \vdash H'_a \) iff either \( H' = G' \) or \( H'_a = G' \) or \( H'_a \in K' \), \( K' \) is not a countermodel for \( \varphi \). Let \( \varphi' \) be an extension of \( \varphi \) to \((G', K', R')\); i.e., \( \varphi'(P, H) = \varphi(P, H) \) for \( H \in K' \). Then it is easily proved by induction that \( \varphi'(B, H) = \varphi^- (B, H) \) for \( H \in K \) and any subformula \( B \) of \( A \). So \( \varphi'(A, G) = \varphi(A, G) = F \). Since \( G' \vdash G \), \( \varphi'(A, G') = F \). Since \( G' \vdash G' \), \( \varphi'(A, G') = F \). Hence, by Theorem 2, \( \varphi' \) is a m.s., hence \( \varphi' \) is an S2 normal model. Let \( \varphi \) be an extension of \( \varphi \) to \((G', K', R')\); i.e., \( \varphi'(P, H) = \varphi(P, H) \) for \( H \in K' \). Then it is easily proved by induction that \( \varphi'(B, H) = \varphi^- (B, H) \) for \( H \in K \) and any subformula \( B \) of \( A \). So \( \varphi'(A, G) = \varphi(A, G) = F \). Since \( G' \vdash G \), \( \varphi'(A, G') = F \). Since \( G' \vdash G' \), \( \varphi'(A, G') = F \). Hence, by Theorem 2, \( \varphi' \) is a m.s., hence \( \varphi' \) is an S2 model, so that \( \Box A \) is not valid, hence not provable, in S2. Similarly for S2 and S3. (In the case of S3, \( n > 1 \), instead of adjoining a single element \( G' \) to \( K \), we form \( K \) by adjoining a string of new elements \( G_1, G_2, \ldots, G_n, \) and stipulating that \( G_i \vdash G_i \) for \( i = 1, \ldots, n \), \( G_i \vdash G_i \). Further for \( H, H' \in K, \) \( H \vdash H' \) iff \( H \vdash H' \). Form \((G_1, G_2, \ldots, G_n, R')\), where \( R' \) holds only as required above; this is clearly an S2 m.s. If \( \varphi \) extends \( \varphi \) to \((G_1, G_2, \ldots, G_n, R')\), then \( \varphi((\Box)^n A, G_1) = F \) if \( \varphi(A, G) = F \). Q.E.D.

We can now, incidentally, show that the axiomatizations of S2 and S3 given here are in fact equivalent to the original systems of Lewis (Lewis and Langford [59]). Actually, we prove their equivalence to Lemmon’s P2 and P3, respectively; a proof that Lemmon’s systems are equivalent to Lewis’s forms of S2 and S3 has been given in Lemmon [57].

The axioms of P2 are the axioms of P2 (E3). In addition to modus ponens, Lemmon’s rules for P2 (P3) are a rule of inference. To derive the first rule in S2 (S3), as it has been formulated here, observe that if \( A \) is a tautology or an axiom, \( \vdash \Box A \); and a rule \( (A \Box B) / (A \Box B) \). To derive the second rule in S2 (S3), as it has been formulated here, observe that if \( \vdash \Box A \) is a tautology or an axiom of P2 (P3), it is a theorem of E2 (E3); so, by Theorem 6, \( \Box A \) is a theorem of S2 (S3). For the second rule, observe that if \( \vdash (A \Box B) \) in S2 (S3), by Theorem 6 \( \vdash A \Box B \) in E2 (E3), hence by (Eb) \( \vdash \Box (A \Box B) \) in E2 (E3), and hence by Theorem 6 \( \vdash (A \Box B) \) in S2 (S3). So S2 and S3 contain P2 (P3); the converse inclusion is trivial.

We notice that, as in Kripke [63], the tableaux lead to a decision procedure for the propositional calculi considered. In fact, in S2, a formula \( A \) of degree \( m \) is valid or has a finite tree countermodel, in which each branch of the tree is of length \( <m \). The proof is like that for M in Kripke [63], in E3 and S3, we prove, similarly to the proof for S4 in Kripke [63], that a formula \( A \) is valid or has both a finite countermodel and a (possibly infinite) denumerable tree countermodel.

\[ \text{Theorem 7.} \quad \text{If } A \text{ is a formula of degree } < m, \text{ then } \vdash A \text{ in } S2^{m+1} \text{ if and only if } \vdash A \text{ in } M. \]

\[ \text{Proof.} \quad \text{The "only if" part is trivial. If } A \text{ is not provable in } S2^{m+1}, \text{ then it has, being of degree } < m, \text{ a finite tree countermodel } \varphi \text{ in which each branch is of length } < m, \text{ and hence if } (G, K, R) \text{ is the associated model structure, every } H \in K \text{ has depth } < m. \text{ Since } (G, K, R) \text{ is an } S2^{m+1} \text{ model structure, this shows that every } H \in K \text{ is normal, and hence that } (G, K, R) \text{ is an M m.s. But then } \varphi \text{ is a countermodel to } A \text{ in } M. \]

\[ \text{Remark.} \quad \text{By use of semantic tableaux and a modified proof, we can improve } S2^{m+1} \text{ in this result to } S2^m. \text{ The latter is a "best possible" result; for no formula } \Box^n (A \Box A) \text{ (} m > 1 \text{) can be provable in } S2^{m+1} \text{ (otherwise, by Theorem 6, } \vdash (A \Box A) \text{ in } E2). \]

\[ \text{Corollary.} \quad M \text{ has no axiomatization with only finitely many axiom schemata, and with modus ponens as sole rule of inference.} \]

\[ \text{Proof.} \quad \text{If it had such an axiomatization, let } m \text{ be the maximum modal degree of the schemata. By Theorem 7, } \text{ every axiom would be provable in } S2^{m+1}, \text{ and since modus ponens is a rule of } S2^{m+1}, \text{ M would be contained in } S2^{m+1}. \text{ But } \Box^m (A \Box A) \text{ (} m > 1 \text{) is not a theorem of } S2^{m+1}, \text{ as shown in the remark just made; so the reductio is complete.} \]

\[ \text{This corollary has long been a widespread unproved conjecture.} \]

Finally, notice that, as in Kripke [63], given a m.s. \((G, K, R)\) we can define a proposition \( \varphi \) as a mapping of elements of \( K \) into \( \{T, F\} \), or, equivalently, as a subset of \( K \). Given propositions \( \varphi \) and \( \sigma \), define \( \varphi \land \sigma \), \( \Diamond \), as in Kripke [63]; \( \Box \varphi \) is now defined by \( \Box \varphi = T \) iff \( \varphi \in K \) and \( \varphi(H') = F \) for all \( H' \in K \) such that \( H \vdash H' \); otherwise \( \Box \varphi = F \).

As in Kripke [63], we get matrices for S2, E3, S3 from model structures for these systems. Let \( K = K, H = H \), and let \( 1 = \{G, H, 2, = G, 3 = H, 4 = 0\} \), 4 = \( \emptyset \), set, taking propositions as subsets of \( K \), as in Kripke

\[ \text{5 Strictly speaking, a single schema like } \Box (A \Box A) \text{ can have an arbitrarily large modal degree, depending on the modal degree of } A. \text{ We make the convention that the degree of a schema is to be computed as if the schematic letters therein are atomic formulae; thus } \Box (A \Box A) \text{ has degree 1. Note that if a schema has degree } m \text{ in this sense, we can replace distinct schematic letters by distinct propositional variables to obtain a formula } B \text{ of degree } m \text{ in the ordinary sense. Hence, by Theorem 7, } \vdash B \text{ in } S2^{m+1}. \text{ But since } S2^{m+1} \text{ contains the rule of substitution as a derived rule, the original schema thus is derivable in } S2^{m+1}. \]
Section 5.3, we can prove the following theorem on the contrary to Rallden's results.

It should be mentioned that, by a proof like that in Kripke [63], p. 93. If \( G \) \( R \) \( G \) \( H \), but neither \( H \) \( R \) \( H \), \( G, K, R \) is an \( S3 \) m.s., and the matrix in terms of 1, 2, 3, 4, again with 1 and 2 as designated values, coincides with Group I of Lewis-Langford [59].

Notice that it is impossible to obtain infinite characteristic matrices for \( S2^2 \), \( E3 \), and \( S3 \) of the types obtained in Kripke [63] for the systems considered there; for such characteristic matrices would be normal, contrary to Haldén's results.

If we define an \( S6 \) (\( S2 + \Box \Diamond A^6 \)), \( S7 = (-S3 + \Box \Diamond A^6), S8 = (-S3 + \Box \Diamond A^6) \). Theorem 2 implies that \( S3 \rightarrow S4 \land S7, S2 \rightarrow S2 \land S6, \) since \( \Box \Diamond A \lor \Box \Diamond B \) in \( S3 \) and \( \Box \Diamond A \lor \Box \Diamond B \) in \( S2 \). If we define an \( S6 \) (\( S7 \)) m.s. to be an \( S2 \) (\( S3 \)) m.s. (\( G, K, R \)) with at least one \( H \in K \) of depth 1 which is not normal (i.e., to be an \( S2 \) (\( S3 \)) m.s. which is not an \( S2^2 \) (\( S3^2 \)) m.s.), and an \( S8 \) m.s. to be an \( S3 \) m.s. in which for every normal \( H \in K \) there is an \( H' \in K \), such that \( H \rightarrow H' \), and which is not normal, we can derive completeness theorems for the resulting systems.

Finally, if we drop quasi-reflexivity from the requirements on \( R \), and simply define a model structure to be a quadruple \( (G, K, R, N) \), where \( N \) is the set of normal elements, we get a model theory for Lemmon's D2. If \( R \) is required to be transitive, the theory works for D3.

1. Introduction. The Fraenkel-Mostowski (FM) method as introduced by Fraenkel [22, 28, 28a, 32, 35, 37] and Mostowski [38, 38a, 39] turned out to be very useful in proving the independence of the axiom of choice and related statements. Until the recent appearance of Cohen [63, 64] it was the only method available for obtaining such results. The major deficiency of the FM method is that it does not apply to "ordinary" set theories, i.e., to set theories in which all the elements are sets and in which the axiom of foundation holds. The FM method is applied to the following two types of set theories. The first type, the one which Fraenkel and Mostowski originally dealt with, are set theories which admit the existence of urelements, i.e., elements which are not sets. These set theories are obtained from the "ordinary" set theories by what essentially amounts to a weakening of the axiom of extensionality. The second type, to which this method was applied by Mendelson [56], Shoenfield [55] and Specker [57], are set theories which permit the existence of unfounded sets, i.e., elements, which are not sets. These set theories are obtained from the "ordinary" set theories by the same proofs of independence were obtained in both types of set theories, except for a proof of the independence of the axiom of foundation from some other statements in Mendelson [58]. Since all other known proofs are carried out in, practically, the same way in both types of set theories we shall henceforth mention only the first type, tacitly assuming that everything applies to the second type as well.

The new method of Cohen [63, 64] for proving the independence of the axiom of choice and related statements is stronger than the FM method in that it applies also to "ordinary" set theories. However, independence

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