Some Remarks on the Semantics of Basic Modal Logic

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For questions of truth preservation in basic modal logic, one can use tableaux (which we will discuss this week). But, for questions of validity preservation, one must reason directly using the definition of validity. In this handout, I will go through some of the arguments I gave in class last week, concerning validity preservation in basic modal logic. First, let me remind you of the crucial semantical definitions.

- For any interpretation $I = \langle W, R, v_w \rangle$, $v_w(\Box p) = 1$ iff for all $w' \in W$ such that wRw', $v_{w'}(p) = 1$. In other words, $\Box p$ is true at w (on I) iff p is true at *all* worlds $w' \in W$ that are accessible from w.

- For any interpretation $I = \langle W, R, v_w \rangle$, $v_w(\Diamond p) = 1$ iff there *exists* a $w' \in W$ such that wRw' and $v_{w'}(p) = 1$. That is, $\Diamond p$ is true at w (on I) iff p is true at *some* world $w' \in W$ that's accessible from w.

  - It follows immediately from these definitions that $\Box p \dashv\vDash \neg\Diamond\neg p$, and $\Diamond p \dashv\vDash \neg\Box\neg p$. This is because $\Box p$ is true iff a condition of the form $(\forall w)\phi w$ holds, while $\Diamond p$ is true iff $(\exists w)\phi w$ holds. And, as you'll recall from 12A, we have: $(\forall w)\phi w \dashv\vDash \neg(\exists w)\neg\phi w$, and $(\exists w)\phi w \dashv\vDash \neg(\forall w)\neg\phi w$.

- $\Sigma \vDash p$ iff for *every* interpretation $I = \langle W, R, v_w \rangle$ and *every* world $w \in W$:

  If $v_w(s) = 1$ for all $s \in \Sigma$, then $v_w(p) = 1$.

  In other words, $\Sigma \vDash p$ iff the inference from Σ to p preserves *truth* in *all* worlds of *all* interpretations.

- $\vDash p$ is just a special case in which $\Sigma = \varnothing$. Hence, $\vDash p$ iff p is true at *all* worlds of *all* interpretations.

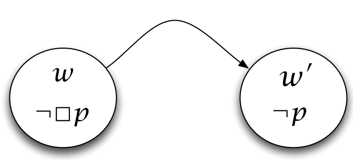
With these definitions in mind, here are some facts about validity preservation that I discussed last week.

1. $\vDash p \implies \vDash \Box p$.

   Proof. To establish (1), I will prove its *contrapositive*: $\nvDash \Box p \implies \nvDash p$. Suppose $\nvDash \Box p$. Then, there must exist a world w in an interpretation $I = \langle W, R, v_w \rangle$, such that $v_w(\Box p) = 0$. Thus, $v_w(\neg\Box p) = 1$. Hence, $v_w(\Diamond\neg p) = 1$. So, there must exist a world $w' \in W$ (of I) such that wRw' and $v_{w'}(\neg p) = 1$. Thus, $v_{w'}(p) = 0$, for *some* world w' on *some* interpretation I, which implies $\nvDash p$. □

2. $\vDash \Box p \implies \vDash p$.

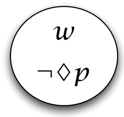
   Proof. Again, I will prove the *contrapositive* of (2): $\nvDash p \implies \nvDash \Box p$. Suppose $\nvDash p$. Then, there will exist a world w' in an interpretation I such that $v_{w'}(p) = 0$. Moreover, because p is not logically true, there will *also* exist some interpretations I' in which *w' is accessible from some world w*. And, the existence of *these* sorts of interpretations are what ensure that $\nvDash \Box p$. Here is what such an I' "looks like":

   More formally, we would say: $I' = \langle W, R, v_w \rangle$, where $W = \{w, w'\}$, $R = \{\langle w, w' \rangle\}$, and $v_{w'}(p) = 0$ [hence, $v_w(\Box p) = 0$]. We know such I's exist, because $\nvDash p$ implies that $\neg p$ is logically possible. Thus, there will exist logically possible/consistent worlds w and w' like these in *some* I's such as these. □

3. $\vDash p \not\Rightarrow \vDash \Diamond p$. That is, there are some sentences p such that $\vDash p$, but $\nvDash \Diamond p$.

Proof. Let p be any classical tautology [*e.g.*, $A \supset A$]. Then, $\vDash p$, since all classical tautologies are true at all worlds of all interpretations of basic modal logic [*e.g.*, $v_w(A \supset A) = 1$ for all worlds w in all interpretations I]. But, we do *not* have $\vDash \Diamond p$, since *all* $\Diamond$-statements will be *false* at any "dead-end" world. *E.g.*, let $I = \langle W, R, v_w \rangle$, where $W = \{w\}$, and $R = \varnothing$. Then, we will have $v_w(\Diamond p) = 0$, for *all* sentences p, since *there are no worlds (in W) that are accessible from w*. We can picture I, as follows:



The existence of such interpretations ensures that $\nvDash \Diamond p$, for *all* sentences p of basic modal logic. □

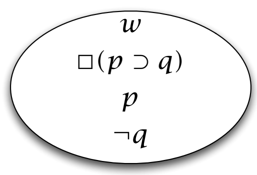
4. $\vDash \Diamond p \Longrightarrow \vDash p$.

   Proof. This direction holds *vacuously*, since [as we just established in (3), above], $\nvDash \Diamond p$, for *all* sentences p of basic modal logic. Remember, because our meta-theoretic conditional '$\Longrightarrow$' is classical, it is always true, so long as its antecedent is always false. And, (4) *always* has a false antecedent. □

5. *Define* $p \strictif q \overset{\text{def}}{=} \Box(p \supset q)$. We call '$\strictif$' a *strict conditional*. In basic modal logic, the strict conditional has some strongly *non-classical* properties. Here are some of these properties:

   (a) *Modus Ponens* is *not* truth preserving for '$\strictif$' in basic modal logic. That is: $\{p, p \strictif q\} \nvDash q$.

   Proof. One can use "dead-end" worlds to construct counterexamples to *Modus Ponens* for '$\strictif$' in basic modal logic. *E.g.*, let $I = \langle W, R, v_w \rangle$, where $W = \{w\}$, $R = \varnothing$, $v_w(p) = 1$, and $v_w(q) = 0$.



   Here, $\Box(p \supset q)$ will be true at w (on I) because *all* $\Box$-statements are (*vacuously*) true at w. We can just use the tableau method for basic modal logic to prove this. Here's how it would look:

$$\begin{array}{c} \Box(p \supset q), 0 \\ p, 0 \\ \neg q, 0 \end{array}$$

   In this open tableau, world "0" is just like our "dead-end" world w in our I pictured above. □

   (b) *Modus Ponens is validity* preserving for '$\strictif$'. That is: *If* $\vDash p$ and $\vDash p \strictif q$, *then* $\vDash q$.

   Proof. Suppose $\vDash p$ and $\vDash \Box(p \supset q)$. Then, by (2) above, we have $\vDash p$ and $\vDash p \supset q$. Therefore, by the validity-preservingness of *Modus Ponens* for '$\supset$', we have $\vDash q$, as desired. □

   (c) Recall the five "problematic validities" Priest discussed for the material conditional '$\supset$':

      i. $q \vDash p \supset q$.
      ii. $\neg p \vDash p \supset q$.
      iii. $(p \,\&\, q) \supset r \vDash (p \supset r) \vee (q \supset r)$.
      iv. $(p \supset q) \,\&\, (r \supset s) \vDash (p \supset s) \vee (r \supset q)$.
      v. $\neg(p \supset q) \vDash p$.

   None of these inferences is truth-preserving for the basic strict conditional '$\strictif$'! *Prove* this with tableaux! Which of (i)–(v) *are validity-preserving* for the the basic strict conditional '$\strictif$'? *Prove* it!

   (d) $(p \,\&\, q) \strictif r \nvDash p \strictif (q \strictif r)$. *I.e.*, *Exportation* is *in*valid for '$\strictif$'. *Prove* this with a tableau! In fact, Exportation isn't even *validity* preserving for the basic strict conditional '$\strictif$'! *Prove* this!