k-Validity *vs* Validity in Q: A formula of Q that is k-valid *for all* k, but *not* valid

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Consider the following three formulas of Q [where, as always, $p \wedge q \overset{\text{def}}{=} \sim(p \supset \sim q)$, and $\bigvee p \overset{\text{def}}{=} \sim \bigwedge \sim p$]:

$$p \overset{\text{def}}{=} \bigwedge x' \bigwedge x'' \bigwedge x'''[(F^{**\prime}x'x'' \wedge F^{**\prime}x''x''') \supset F^{**\prime}x'x''']$$

- In more standard notation: $p \overset{\text{def}}{=} (\forall x)(\forall y)(\forall z)[(Rxy \wedge Ryz) \supset Rxz]$.

$$q \overset{\text{def}}{=} \bigwedge x' \bigvee x'' F^{**\prime}x'x''$$

- In more standard notation: $q \overset{\text{def}}{=} (\forall x)(\exists y)Rxy$.

$$r \overset{\text{def}}{=} \bigvee x' F^{**\prime}x'x'$$

- In more standard notation: $r \overset{\text{def}}{=} (\exists x)Rxx$.

Informally, p asserts that the 2-place relation $F^{**\prime}$ (R) is *transitive*, q asserts that $F^{**\prime}$ (R) is *serial*, and r asserts that there is some object that bears the relation $F^{**\prime}$ (R) *to itself*. Now, consider the following complex statement, constructed out of p, q, and r:

$$A \overset{\text{def}}{=} (p \wedge q) \supset r$$

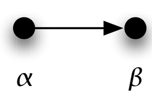
Claim A asserts that *if* $F^{**\prime}$ (R) is transitive and serial, *then* some object bears $F^{**\prime}$ (R) *to itself*.

Fact. A is k-valid *for all* (finite) k, but A is *not* valid [$\not\vDash_Q A$].

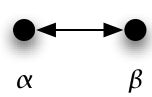
Proof. First, we will show (informally) that A is k-valid, *for all* k. Our (informal) argument will involve showing that A is true on all 1-element interpretations, and all 2-element interpretations, and ..., and all k-element interpretations, for all k. We will do this by arguing (informally) that we cannot make A *false* on any k-element interpretation. And, since A is a closed formula, it must either be true or false on each interpretation of Q. Thus, it will follow that A is true on all k-element interpretations of Q, *for all* k.

Let's think about 1-element interpretations I_1 first. In order to make A false on *any* interpretation I, we would need to make p and q both true, and r false on I (these are just the falsity-conditions for $\supset$). And, to make r *false* on a 1-element interpretation I_1, we need to ensure that its single element α is such that $\sim R\alpha\alpha$. But, we also need to ensure that q is *true* on I_1. Thus, we need there to be *some* element β of I_1 such that $R\alpha\beta$ is *true* on I_1. Because there is only one element in the interpretation, we cannot make q true while r is false. This shows (already, and without even considering that p must *also* be true on I_1 in order to make A false on I_1) that *there can be no 1-element interpretation* I_1 *on which* A *is false*. In other words, we have just shown that the formula A *is 1-valid* (indeed, we've even shown that $q \supset r$ is 1-valid).

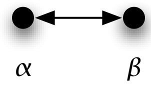
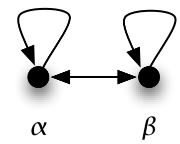
Next, we need to argue that we cannot make A false on a *2*-element interpretation I_2 either. So far, we have two elements α and β, with the following structure (arrows represent R-relations):



Now, because we need r to *remain false* on I_2, we must have $\sim R\beta\beta$ on I_2. And, because we need q to *remain true* on I_2, we need there to be *some* γ such that $R\beta\gamma$ is true on I_2. The only way to do this (without adding yet another element to our interpretation I_2) is to *try* to make $R\beta\alpha$ true on I_2, which would yield the following (symmetric) structure:

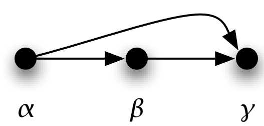


This structure I_2 *is* one on which q is true and r is false. But, now we have a problem with ensuring that p is *true* on I_2. If the truth of p were enforced here, then we would end-up with both $R\alpha\alpha$ and $R\beta\beta$. That is:

Transitivity of R and  entails: 

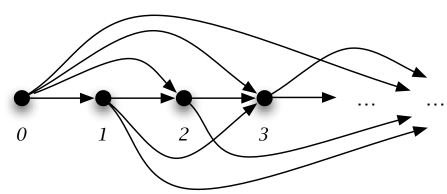
To see this, just look at the instance of p, where $x' = \alpha$, $x'' = \beta$, and $x''' = \alpha$ (and then the instance of p, where $x' = \beta$, $x'' = \alpha$, and $x''' = \beta$). Therefore, it is impossible to make A false on an interpretation containing only two elements I_2. That is, we have just shown that A *is 2-valid*.

Perhaps we can make A false on an interpretation with *three* elements I_3? We have just seen that in order to make p true, while q is true and r is false, we need to add a *third* object γ to I_2, which would yield an interpretation I_3 with the following relational structure:



Again, in order to make r false on I_3, we must have $\sim R\gamma\gamma$ on I_3, and in order to make q true on I_3, there must be *some* object δ such that $R\gamma\delta$. We could *try* to satisfy this constraint by forcing either $R\gamma\alpha$ or $R\gamma\beta$ on I_3. But, both of these choices will end-up with the same sort of inconsistency with p and $\sim r$ that we just saw in the previous ($k = 2$) case, with the introduction of object β. That is, if we enforce $R\gamma\alpha$, then p will entail $R\alpha\alpha$ and $R\gamma\gamma$ (hence r will be *true*), and if we enforce $R\gamma\beta$, then p will entail $R\beta\beta$ and $R\gamma\gamma$ (and r will be *true*). And, this frustrating story will repeat itself, no matter how large the (finite) domain of I_k is allowed to be — *A cannot be false on any interpretation of (finite) size k*. $\therefore$ A is k-valid, for all k.

The second thing we need to demonstrate is that A is *not* valid [*i.e.*, $\nvDash_Q A$]. Remember, just because all *finite* interpretations have a certain property, it doesn't follow that all *infinite* interpretations must also have that property (just think about the property of *being finite*!). So, just because A is true on all finite interpretations (as the informal argument above shows), it doesn't follow that A is true on all infinite interpretations as well. And, in fact, *A is false on some infinite interpretations*. Let I_∞ be an interpretation of Q in which $D = \mathbb{N}$, and $F^{**\prime}$ gets assigned the relation $Rxy \overset{\text{def}}{=} x < y$ by I_∞. Here's a "picture" of I_∞:



A is *false* on I_∞. To demonstrate this, we need to show that p and q are both true on I_∞, but r is false on I_∞. It is easy to see that both p and q are true on I_∞, since the less-than relation is clearly both transitive and serial on the natural numbers. And, it is also clear that r is false on I_∞, since *no* natural number is less than itself. This completes the (informal) proof that A is k-valid, for all (finite) k, *but* A is invalid . □

Addendum: Can you give more *rigorous* proofs of these two claims about A? Presumably, the first claim would proceed *via* induction on the size k of candidate interpretations I_k. We have already established the basis ($k = 1$) case, above. The inductive hypothesis would be that A is false on all interpretations I_j of size $1 \leq j < k$. And, from this assumption, the goal would be to prove that A is false on interpretations I_k of size equal to k. You can see how that argument is going to run, just by thinking about how we were led into frustrating inconsistencies above, when we tried to make A true by adding one element to an interpretation I_j on which A was false. For the second claim, can this one be proved (in an illuminating way) by induction? Or does it involve properties so basic to the less-than relation (over natural numbers) that it would be difficult to see how an illuminating "inductive proof" would go?