Consider the following three formulas of Q [where, as always, \( p \land q \equiv \neg (p \lor \neg q) \), and \( \lor p \equiv \neg \land \neg p \)]:

\[
q \equiv \land x' \land x'' \land x'''[(F**'x'x'' \land F**'x''x''') \lor F**'x'x''']
\]

- In more standard notation: \( q \equiv (\forall x)(\forall y)(\forall z)[(Rxy \land Ryz) \lor Rxz] \).

\[
r \equiv \land x'F**'x'x''
\]

- In more standard notation: \( r \equiv (\exists x)Rxx \).

Informally, \( p \) asserts that the 2-place relation \( F**' (R) \) is transitive, \( q \) asserts that \( F**' (R) \) is serial, and \( r \) asserts that there is some object that bears the relation \( F**' (R) \) to itself. Now, consider the following complex statement, constructed out of \( p, q \), and \( r \):

\[ A \equiv (p \land q) \lor r \]

Claim \( A \) asserts that if \( F**' (R) \) is transitive and serial, then some object bears \( F**' (R) \) to itself.

**Fact.** \( A \) is \( k \)-valid for all (finite) \( k \), but \( A \) is not valid \([\not\models_Q A] \).

**Proof.** First, we will show (informally) that \( A \) is \( k \)-valid, for all \( k \). Our (informal) argument will involve showing that \( A \) is true on all 1-element interpretations, and all 2-element interpretations, and ..., and all \( k \)-element interpretations, for all \( k \). We will do this by arguing (informally) that we cannot make \( A \) false on any \( k \)-element interpretation. And, since \( A \) is a closed formula, it must either be true or false on each interpretation of \( Q \). Thus, it will follow that \( A \) is true on all \( k \)-element interpretations of \( Q \), for all \( k \).

Let’s think about 1-element interpretations \( I_1 \) first. In order to make \( A \) false on any interpretation \( I \), we would need to make \( p \) and \( q \) both true, and \( r \) false on \( I \) (these are just the falsity-conditions for \( \lor \)). And, to make \( r \) false on a 1-element interpretation \( I_1 \), we need to ensure that its single element \( \alpha \) is such that \( \neg R\alpha\alpha \). But, we also need to ensure that \( q \) is true on \( I_1 \). Thus, we need there to be some element \( \beta \) of \( I_1 \) such that \( R\alpha\beta \) is true on \( I_1 \). Because there is only one element in the interpretation, we cannot make \( q \) true while \( r \) is false. This shows (already, and without even considering that \( p \) must also be true on \( I_1 \) in order to make \( A \) false on \( I_1 \)) that there can be no 1-element interpretation \( I_1 \) on which \( A \) is false. In other words, we have just shown that the formula \( A \) is 1-valid (indeed, we’ve even shown that \( q \lor r \) is 1-valid).

Next, we need to argue that we cannot make \( A \) false on a 2-element interpretation \( I_2 \) either. So far, we have two elements \( \alpha \) and \( \beta \), with the following structure (arrows represent \( R \)-relations):

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\alpha \rightarrow \beta
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Now, because we need \( r \) to remain false on \( I_2 \), we must have \( \neg R\beta\beta \) on \( I_2 \). And, because we need \( q \) to remain true on \( I_2 \), we need there to be some \( y \) such that \( R\beta y \) is true on \( I_2 \). The only way to do this (without adding yet another element to our interpretation \( I_2 \)) is to try to make \( R\beta\alpha \) true on \( I_2 \), which would yield the following (symmetric) structure:

```
\alpha \rightarrow \beta
```

\[ \alpha \rightarrow \beta \]
Transitivity of $R$ and $\alpha, \beta$ entails:

![Diagram showing transitivity](image)

To see this, just look at the instance of $p$, where $x' = \alpha$, $x'' = \beta$, and $x''' = \alpha$ (and then the instance of $p$, where $x' = \beta$, $x'' = \alpha$, and $x''' = \beta$). Therefore, it is impossible to make $A$ false on an interpretation containing only two elements $I_2$. That is, we have just shown that $A$ is 2-valid.

Perhaps we can make $A$ false on an interpretation with three elements $I_3$? We have just seen that in order to make $p$ true, while $q$ is true and $r$ is false, we need to add a third object $y$ to $I_2$, which would yield an interpretation $I_3$ with the following relational structure:

![Diagram showing additional object](image)

Again, in order to make $r$ false on $I_3$, we must have $\neg R_{yy}$ on $I_3$, and in order to make $q$ true on $I_3$, there must be some object $\delta$ such that $R_{y\delta}$. We could try to satisfy this constraint by forcing either $R_{y\alpha}$ or $R_{y\beta}$ on $I_3$. But, both of these choices will end up with the same sort of inconsistency with $p$ and $\neg r$ that we just saw in the previous ($k = 2$) case, with the introduction of object $\beta$. That is, if we enforce $R_{y\alpha}$, then $p$ will entail $R_{\alpha\alpha}$ and $R_{yy}$ (hence $r$ will be true), and if we enforce $R_{y\beta}$, then $p$ will entail $R_{\beta\beta}$ and $R_{yy}$ (and $r$ will be true). And, this frustrating story will repeat itself, no matter how large the (finite) domain of $I_k$ is allowed to be — $A$ cannot be false on any interpretation of (finite) size $k$. \(\vdash A \text{ is } k\text{-valid, for all } k\).

The second thing we need to demonstrate is that $A$ is not valid [i.e., $\not\models \alpha \ A$]. Remember, just because all finite interpretations have a certain property, it doesn’t follow that all infinite interpretations must also have that property (just think about the property of being finite!). So, just because $A$ is true on all finite interpretations (as the informal argument above shows), it doesn’t follow that $A$ is true on all infinite interpretations as well. And, in fact, $A$ is false on some infinite interpretations. Let $I_\infty$ be an interpretation of $Q$ in which $D = \mathbb{N}$, and $F^{**}$ gets assigned the relation $R_{xy} \equiv x < y$ by $I_\infty$. Here’s a “picture” of $I_\infty$:

![Diagram showing infinite structure](image)

$A$ is false on $I_\infty$. To demonstrate this, we need to show that $p$ and $q$ are both true on $I_\infty$, but $r$ is false on $I_\infty$. It is easy to see that both $p$ and $q$ are true on $I_\infty$, since the less-than relation is clearly both transitive and serial on the natural numbers. And, it is also clear that $r$ is false on $I_\infty$, since no natural number is less than itself. This completes the (informal) proof that $A$ is $k$-valid, for all (finite) $k$, but $A$ is invalid.

**Addendum:** Can you give more rigorous proofs of these two claims about $A$? Presumably, the first claim would proceed via induction on the size $k$ of candidate interpretations $I_k$. We have already established the basis ($k = 1$) case, above. The inductive hypothesis would be that $A$ is false on all interpretations $I_j$ of size $1 \leq j < k$. And, from this assumption, the goal would be to prove that $A$ is false on interpretations $I_k$ of size equal to $k$. You can see how that argument is going to run, just by thinking about how we were led into frustrating inconsistencies above, when we tried to make $A$ true by adding one element to an interpretation $I_j$ on which $A$ was false. For the second claim, can this one be proved (in an illuminating way) by induction? Or does it involve properties so basic to the less-than relation (over natural numbers) that it would be difficult to see how an illuminating “inductive proof” would go?