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EXTENDIBLE SENTENTIAL CALCULUS

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1. Introduction. The system of sentential calculus presented below has peculiarities which may be of some interest. It is complete in the sense that every classical (two-valued) tautology is provable. In spite of this it is not Post complete (nor is it absolutely complete),[1] i.e., one can add to the system an unprovable formula without proving a sentential variable as a theorem (or without making every formula a theorem). Thus the system may be extended by adjoining an unprovable formula without making the system inconsistent in the Post (or in absolute) sense. There are many distinct extensions of this kind. For every formula α_1 that is not a theorem and which adjoined to the system does not make the system Post (or absolutely) inconsistent there is a formula α_2 such that adjunction of α_2 to the original system leaves α_1 unprovable. An infinite sequence of admissible formulas may be exhibited such that each of them when adjoined to the original system does not provide a deduction of the preceding one. The system is based on a finite number of axiom schemes (or a finite number of axioms and the rule of substitution) and a finite number of rules of inference of a specific kind. The rules allowed here are, so to speak, translatable into formulas of the sentential calculus. Thus the rule 'if $\alpha \supset \beta$ and $\beta \supset \gamma$ are theorems, so is $\alpha \supset \gamma$' has as its translation '$(p \supset q) \supset ((q \supset r) \supset (p \supset r))$', and '$p \supset (q \supset p)$' is a translation of the rule 'if α is a theorem, then any conditional with α as its consequent is a theorem.' The notation '$\alpha_1, \alpha_2, \ldots, \alpha_n \Rightarrow \beta$' shall be used for 'if $\alpha_1, \alpha_2, \ldots, \alpha_n$ are theorems, so is β'; Greek letters, with or without subscripts, range over formulas. There are rules of inference that are not translatable into sentential formulas, for instance the rule of substitution or the rule 'if α is a tautology, then α is a theorem.' More precisely, the only rules, besides the rule of substitution, allowed here are of the form '$A_1, A_2, \ldots, A_i \Rightarrow B$' where $A_1, A_2, \ldots, A_i$ and B contain Greek letters and if one replaces the Greek letters by sentential variables, and the arrows and the commas by the sign of conditional, one obtains (after affixing proper parentheses) a formula of the sentential calculus. The results of this paper are not completely trivial merely because the rules are in the indicated form; the restriction on the rules of inference has the effect of excluding trivially extendible systems. Had one not required it, one could give as an example of an extendible system the system based on: (a) every tautology is a theorem, (b) the formula '$q \supset p$' is a theorem.

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[1] For definitions of Post and absolute completeness and consistency see [1] pp. 108–110.

The system of sentential calculus constructed below admits of "non-standard" models; although all and only classical tautologies are provable, there are non-trivial many-valued models of the system. It has a further peculiarity. Modus ponens (the rule of detachment for conditionals) is neither among the primitive rules of inference nor among the derivable rules of inference (that is, rules such that each application of them can be uniformly replaced by a specific finite sequence of applications of the primitive rules). Modus ponens is, however, admissible in the sense that if it is adjoined to the list of rules of the system, there is no resulting change in the set of theorems. On the other hand for no extended system is modus ponens admissible in this sense. The opinion, sometimes expressed, that a complete system of tautologies constitutes an adequate characterization of valid inferences of the sentential kind is shown to be unjustified. For, usually, when a theorem of the sentential calculus is proven, then the rule of which this theorem is a translation is considered established and the completeness of the sentential calculus indicates that all the valid rules of the kind described above are derivable. Here, however, the system is complete and yet some valid rules of this kind are not derivable (e.g., modus ponens; $\sim\sim\alpha \Rightarrow \alpha$; α, $\sim\alpha \Rightarrow \beta$). Let the subscript '*mp*' indicates that modus ponens is used and the subscript '*nmp*' that modus ponens is not used. The notation '$\alpha_1, \alpha_2, \ldots, \alpha_n \vdash \beta$' is often employed for 'β is derivable from $\alpha_1, \alpha_2, \ldots, \alpha_n$.' The difference between '$\alpha_1, \alpha_2, \ldots, \alpha_n \Rightarrow \beta$' and '$\alpha_1, \alpha_2, \ldots, \alpha_n \vdash \beta$' is that in the former notation it is essential that $\alpha_1, \alpha_2, \ldots, \alpha_n$ be theorems whereas in the latter it is immaterial whether they are theorems or not. '$\vdash$' asserts only derivability and not provability as '$\Rightarrow$' does. Now, a result of this paper may be phrased: there is a system of sentential calculus for which if $\alpha_1, \alpha_2, \ldots, \alpha_n \Rightarrow_{mp} \beta$, then $\alpha_1, \alpha_2, \ldots, \alpha_n \Rightarrow_{nmp} \beta$ but not if $\alpha_1, \alpha_2, \ldots, \alpha_n \vdash_{mp} \beta$, then $\alpha_1, \alpha_2, \ldots, \alpha_n \vdash_{nmp} \beta$.

The first statement reminds one of Gentzen's main theorem. As a matter of fact Gentzen's system LK is also extendible.[2] One can add to LK any formula that contains a logical connective without making the system Post-inconsistent. This follows from a corollary (2.513) to Gentzen's main theorem to the effect that in a derivation without a "cut" there is no way of eliminating a logical connective that appears in a premiss[3]. Thus, from '$\sim\alpha$' (or from '$\alpha \vee \beta$', or '$\alpha \& \beta$', or '$\alpha \supset \beta$') one cannot deduce α. However, Gentzen's LK is not easily comparable with systems considered in this paper. LK is not exactly a system in the same sense as systems constructed here.

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[2] This fact was called to my attention by W. Craig during a discussion concerning the results of this paper.

[3] [2] p. 51.

2. Extendible calculus. Consider

I. $\alpha \supset \beta,\ \beta \supset \gamma \Rightarrow \alpha \supset \gamma$;

II. $\alpha \supset (\beta \supset \gamma),\ \alpha \supset \beta \Rightarrow \alpha \supset \gamma$;

III. $\sim\alpha \supset \beta,\ \sim\alpha \supset \sim\beta \Rightarrow \alpha$;

IV. $\sim(\alpha \supset \beta) \supset \alpha$;

V. $\sim(\alpha \supset \beta) \supset \sim\beta$.

From the axioms of the form IV and V every logically true formula of the sentential calculus (i.e. a formula satisfied by the usual two-valued matrix M_2) follows by a finite number of applications of the rules of inference I, II, and III. This fact is proven in section 3 of this paper.

Equivalently, instead of all cases of IV and V one can state the rule of substitution and two specific axioms:

IV. $\sim(p \supset q) \supset p$ and

V. $\sim(p \supset q) \supset \sim q$.

Every consequence of the axioms is also satisfied by M_3:

$\supset$	1	2	3	$\sim$
*1	1	3	3	2
*2	1	1	1	1
3	1	1	1	1

where 1 and 2 are designated. However, modus ponens ($\alpha \supset \beta,\ \alpha \Rightarrow \beta$) is not valid in M_3. It is not valid in either of the two following senses[4]: first, in the sense that any assignment of values for Greek letters that makes the premisses designated assigns to the conclusion a designated value (take $\alpha = 2,\ \beta = 3$); second, in the sense that in any particular application of the rule, whenever the premisses take the designated values the conclusion takes designated values only (in the case of modus ponens take '$\sim(p \supset p) \supset p$' and '$\sim(p \supset p)$' as premisses and '$p$' as conclusion.) Thus modus ponens is not a derivable rule of inference.

Every negation is satisfied by M_3. Not every formula, however, is satisfied by M_3. Hence to the system based on I—V one can add $\sim\alpha$ as a new axiom scheme without making the system Post (absolutely) inconsistent. Every theorem of the first system is also satisfied by any matrix M_n ($n > 2$) with n as the only undesignated value, and such that when $n = 2m$ or when $n = 2m+1$

$$x \supset y \left.\begin{matrix} = n \text{ if } x \leqq m \text{ and } y > m \\ = 1 \text{ otherwise}\end{matrix}\right.$$

[4] This distinction was drawn in [3].

$$\sim 1 = n-1$$
$$\sim 2 = n$$
$$\sim n = 1$$
$$\sim x = n-x+1 \text{ if } 2 < x \leqq m$$
$$\sim x = n-x \text{ if } m < x \leqq n-1.$$

Any formula α prefixed by $n-2$ (or more) negations is satisfied by M_n ($n > 2$) but a formula composed of a sentential variable preceded by $n-1$ (or less) negations is not satisfied by M_n. E.g., '$\sim\sim\sim p$' is satisfied by M_5 for no matter what 'p', but '$\sim\sim p$' is not satisfied. However, '$\sim\sim\sim p$' is not satisfied by M_6. Addition of $\sim\sim\sim\alpha$ as a new axiom scheme leaves '$\sim\sim p$' unprovable, addition of $\sim\sim\sim\sim\alpha$ leaves '$\sim\sim\sim p$' unprovable, etc. Thus the system admits infinitely many distinct Post consistent extensions. R. Harrop[5] shows that the system does not admit Post consistent extensions consisting of conditionals only; a Post consistent extension must contain formulas that are negations.

The following derived rules of inference and theorem schemes will be useful in the next section. Omission of parentheses is made with the understanding that association is always to the right.

VIa. $\sim(\alpha_1 \supset \alpha_2 \supset \ldots \supset \alpha_n) \supset \alpha_i$,

VIb. $\sim(\alpha_1 \supset \alpha_2 \supset \ldots \supset \alpha_n) \supset \sim(\alpha_{i+1} \supset \ldots \supset \alpha_n)$

for each α_i *such that* $1 \leqq i < n$.

Proof. 1. $\sim(\alpha_1 \supset \alpha_2 \supset \ldots \supset \alpha_n) \supset \alpha_1$ [IV]

2. $\sim(\alpha_1 \supset \alpha_2 \supset \ldots \supset \alpha_n) \supset \sim(\alpha_2 \supset \ldots \supset \alpha_n)$ [V]

Thus the theorem holds for $i = 1$. Suppose that the theorem holds for $i = k < n-1$, then

3. $\sim(\alpha_1 \supset \alpha_2 \supset \ldots \supset \alpha_n) \supset \sim(\alpha_{k+1} \supset \ldots \supset \alpha_n)$ [hp]

4. $\sim(\alpha_{k+1} \supset \ldots \supset \alpha_n) \supset \alpha_{k+1}$ [IV]

5. $\sim(\alpha_1 \supset \alpha_2 \supset \ldots \supset \alpha_n) \supset \alpha_{k+1}$ [I, 3, 4]

6. $\sim(\alpha_{k+1} \supset \ldots \supset \alpha_n) \supset \sim(\alpha_{k+2} \supset \ldots \supset \alpha_n)$ [V]

$\sim(\alpha_1 \supset \ldots \supset \alpha_n) \supset \sim(\alpha_{k+2} \supset \ldots \supset \alpha_n)$ [I, 3, 6];

and the theorem holds also for $i = k+1$. VIb includes the case when the consequence is just $\sim\alpha_n$ ($i = n-2$).

VII. $\alpha_1 \supset \alpha_2 \supset \ldots \supset \alpha_n$ *and* $\beta \supset \alpha_i$ *for each* α_i *such that* $1 \leqq i < n$, $\Rightarrow \beta \supset \alpha_n$.

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[5] At a meeting (January 9, 1958) of the Logic Seminar at the Pennsylvania State University. To him, and to other members of the Seminar, I am indebted for useful discussions of this paper. The first outline of the results of this paper was presented on July 29, 1957 to the Summer Institute for Symbolic Logic at Cornell University. A short abstract [4] contains the main result.

Proof. The proof is by induction on the length of the first premise. If $n = 2$ and $\alpha_1 \supset \alpha_2$, $\beta \supset \alpha_1$, then $\beta \supset \alpha_2$ by I. Suppose VII holds for $n \leqq k$ and suppose

1. $\alpha_1 \supset \alpha_2 \supset \ldots \supset \alpha_k \supset \alpha_{k+1}$ and
2. $\beta \supset \alpha_i$ for each α_i such that $1 \leqq i \leqq k$, then
3. $\beta \supset \alpha_k \supset \alpha_{k+1}$ [1, 2, hp]

$\beta \supset \alpha_{k+1}$ [II, 3, 2]

VIII. $\alpha_1 \supset \alpha_2 \supset \ldots \supset \alpha_n \Rightarrow \alpha_1 \supset \alpha_2 \supset \ldots \supset \alpha_{n-1} \supset \beta \supset \alpha_n$.

Proof. 1. $\alpha_1 \supset \alpha_2 \supset \ldots \supset \alpha_n$

2. $\sim(\alpha_1 \supset \alpha_2 \supset \ldots \supset \alpha_{n-1} \supset \beta \supset \alpha_n) \supset \alpha_i$ for each α_i, $1 \leqq i < n$ [VIa]

3. $\sim(\alpha_1 \supset \alpha_2 \supset \ldots \supset \alpha_{n-1} \supset \beta \supset \alpha_n) \supset \sim\alpha_n$ [VIb]
4. $\sim(\alpha_1 \supset \alpha_2 \supset \ldots \supset \alpha_{n-1} \supset \beta \supset \alpha_n) \supset \alpha_n$ [VII, 1, 2]

$\alpha_1 \supset \alpha_2 \supset \ldots \supset \alpha_{n-1} \supset \beta \supset \alpha_n$ [III, 4, 3]

IX. $\alpha_1 \supset \alpha_2 \supset \ldots \supset \alpha_{n-1} \supset \alpha_n \supset \alpha_{n+1}$, $\alpha_1 \supset \alpha_2 \supset \ldots \supset \alpha_{n-1} \supset \alpha_{n+1} \supset \alpha_{n+2} \Rightarrow \alpha_1 \supset \alpha_2 \supset \ldots \supset \alpha_{n-1} \supset \alpha_n \supset \alpha_{n+2}$.

Proof. 1. $\alpha_1 \supset \alpha_2 \supset \ldots \supset \alpha_{n-1} \supset \alpha_n \supset \alpha_{n+1}$

2. $\alpha_1 \supset \alpha_2 \supset \ldots \supset \alpha_{n-1} \supset \alpha_{n+1} \supset \alpha_{n+2}$
3. $\sim(\alpha_1 \supset \alpha_2 \supset \ldots \supset \alpha_{n-1} \supset \alpha_n \supset \alpha_{n+2}) \supset \alpha_i$

for each α_i, $1 \leqq i \leqq n$ [VIa]

4. $\sim(\alpha_1 \supset \alpha_2 \supset \ldots \supset \alpha_{n-1} \supset \alpha_n \supset \alpha_{n+2}) \supset \sim\alpha_{n+2}$ [VIb]
5. $\sim(\alpha_1 \supset \alpha_2 \supset \ldots \supset \alpha_{n-1} \supset \alpha_n \supset \alpha_{n+2}) \supset \alpha_{n+1}$ [VII, 1, 3]
6. $\sim(\alpha_1 \supset \alpha_2 \supset \ldots \supset \alpha_{n-1} \supset \alpha_n \supset \alpha_{n+2}) \supset \alpha_{n+2}$ [VII, 3, 5, 2]

$\alpha_1 \supset \alpha_2 \supset \ldots \supset \alpha_{n-1} \supset \alpha_n \supset \alpha_{n+2}$ [III, 6, 4]

X. $\sim\sim\alpha \supset \alpha$.

Proof. 1. $\sim(\sim\sim\alpha \supset \alpha) \supset \sim\sim\alpha$ [IV]

2. $\sim(\sim\sim\alpha \supset \alpha) \sim\alpha$ [V]

$\sim\sim\alpha \supset \alpha$ [III, 2, 1]

XI. $\alpha \supset \alpha$.

Proof. Similar.

XII. $\alpha \supset \beta \supset \alpha$

Proof. 1. $\sim(\alpha \supset \beta \supset \alpha) \supset \alpha$ [IV]

2. $\sim(\alpha \supset \beta \supset \alpha) \supset \sim\alpha$ [VIb]

$\alpha \supset \beta \supset \alpha$ [III, 1, 2]

XIII. $\sim\alpha \supset \alpha \supset \beta$

Proof. 1. $\sim(\sim\alpha \supset \alpha \supset \beta) \supset \sim\alpha$ [IV]

2. $\sim(\sim\alpha \supset \alpha \supset \beta) \supset \alpha$ [VIa]

$\sim\alpha \supset \alpha \supset \beta$ [III, 2, 1]

XIV. $\alpha_1 \supset \alpha_2 \supset \ldots \supset \alpha_n \supset \alpha_n$

Proof. 1. $\sim(\alpha_1 \supset \alpha_2 \supset \ldots \supset \alpha_n \supset \alpha_n) \supset \alpha_n$ [VIa]

2. $\sim(\alpha_1 \supset \alpha_2 \supset \ldots \supset \alpha_n \supset \alpha_n) \supset \sim\alpha_n$ [VIb]

$\alpha_1 \supset \alpha_2 \supset \ldots \supset \alpha_n \supset \alpha_n$ [III, 1, 2]

3. Completeness proof.[6] Suppose that $\delta\bar{\epsilon}\mathrm{Th}$[7]. All formulas may be arranged in a sequence B: $\alpha_1, \alpha_2, \ldots$. One can form class A as the union of all classes A_i such that

$$\mathrm{A}_0 = \hat{\alpha}(\sim\delta \supset \alpha\epsilon\mathrm{Th}),$$
$$\mathrm{A}_{n+1} = \mathrm{A}_n, \text{ if } \alpha_{n+1}\epsilon\mathrm{A}_n,$$
$$\mathrm{A}_{n+1} = \hat{\alpha}(\sim\alpha_{n+1} \supset \alpha\epsilon\mathrm{A}_n), \text{ if } \alpha_{n+1}\bar{\epsilon}\mathrm{A}_n.$$

LEMMA 1. $\mathrm{A}_n \subseteq \mathrm{A}_{n+1}$.

Proof. The classes A_n and A_{n+1} are either the same or

$$\mathrm{A}_n = \hat{\alpha}(\sim\delta \supset \sim\alpha_{i_1} \supset \ldots \supset \sim\alpha_{i_s} \supset \alpha\epsilon\mathrm{Th}),$$

where α_{i_j} is the first formula γ in B such that

$\gamma\bar{\epsilon}\mathrm{A}_{j-1}$, α_{i_s} is α_n, and

$$\mathrm{A}_{n+1} = \hat{\alpha}(\sim\delta \supset \sim\alpha_{i_1} \supset \ldots \supset \sim\alpha_{i_s} \supset \sim\alpha_{i_{s+1}} \supset \alpha\epsilon\mathrm{Th}).$$

For any $\alpha\epsilon\mathrm{A}_n$, $\sim\delta \supset \alpha_{i_1} \supset \ldots \supset \sim\alpha_{i_s} \supset \alpha\epsilon\mathrm{Th}$ and by VIII $\sim\delta \supset \sim\alpha_{i_1} \supset \ldots \supset \sim\alpha_{i_s} \supset \sim\alpha_{i_{s+1}} \supset \alpha\epsilon\mathrm{Th}$, hence $\alpha\epsilon\mathrm{A}_{n+1}$.

L2. *If* $\alpha \supset \beta\epsilon\mathrm{A}$, $\beta \supset \gamma\epsilon\mathrm{A}$, *then* $\alpha \supset \gamma\epsilon\mathrm{A}$.

Proof. 1. $\alpha \supset \beta\epsilon\mathrm{A}_n$

2. $\beta \supset \gamma\epsilon\mathrm{A}_m$

3. $\sim\delta \supset \sim\alpha_{i_1} \supset \ldots \supset \sim\alpha_{i_s} \supset \alpha \supset \beta$ where α_{i_s} is α_n [1]

4. $\sim\delta \supset \sim\alpha_{i_1} \supset \ldots \supset \sim\alpha_{i_t} \supset \beta \supset \gamma$ where α_{i_t} is α_m [2]

By L1 both $\alpha \supset \beta$ and $\beta \supset \gamma$ belong to the higher of A_n and A_m. Suppose that $m \geqq n$; then $t \geqq s$ and

5. $\sim\delta \supset \sim\alpha_{i_1} \supset \ldots \supset \sim\alpha_{i_t} \supset \alpha \supset \beta$ [3]

$\sim\delta \supset \sim\alpha_{i_1} \supset \ldots \supset \sim\alpha_{i_t} \supset \alpha \supset \gamma$ [IX, 5, 4];

[6] This proof is a modification of a completeness proof for ordinary sentential calculus given by Łoś in [5].

[7] 'ϵTH' and '$\bar{\epsilon}$TH' are abbreviations for 'is a theorem' and 'is not a theorem,' respectively; '$\hat{\alpha}$ (. . .)' should be read as 'the class of all formulas α such that . . ..'

and thus $\alpha \supset \gamma \epsilon \mathrm{A}_m$, and $\alpha \supset \gamma \epsilon \mathrm{A}$.

L3. *If* $\alpha \supset (\beta \supset \gamma) \epsilon \mathrm{A}$, $\alpha \supset \beta \epsilon \mathrm{A}$, *then* $\alpha \supset \gamma \epsilon \mathrm{A}$.

Proof. 1. $\alpha \supset (\beta \supset \gamma) \epsilon \mathrm{A}_n$

2. $\alpha \supset \beta \epsilon \mathrm{A}_m$

Let $m \geqq n$, then, as before,

3. $\alpha \supset (\beta \supset \gamma) \epsilon \mathrm{A}_m$.

Suppose that α_{i_s} was the highest formula from the sequence B used in building A_m.

4. $\sim(\sim\delta \supset \sim\alpha_{i_1} \supset \ldots \supset \sim\alpha_{i_s} \supset \alpha \supset \gamma) \supset \sim\delta$ [VIa]
5. $\sim(\sim\delta \supset \sim\alpha_{i_1} \supset \ldots \supset \sim\alpha_{i_s} \supset \alpha \supset \gamma) \supset \sim\delta_{i_j}$ for each j, $1 \leqq j \leqq s$ [VIa]
6. $\sim(\sim\delta \supset \sim\alpha_{i_1} \supset \ldots \supset \sim\alpha_{i_s} \supset \alpha \supset \gamma) \supset \alpha$ [VIa]
7. $\sim(\sim\delta \supset \sim\alpha_{i_1} \supset \ldots \supset \sim\alpha_{i_s} \supset \alpha \supset \gamma) \supset \sim\gamma$ [VIb]
8. $\sim(\sim\delta \supset \sim\alpha_{i_1} \supset \ldots \supset \sim\alpha_{i_s} \supset \alpha \supset \gamma) \supset \alpha \supset \beta$ [VII, 2, 4, 5]
9. $\sim(\sim\delta \supset \sim\alpha_{i_1} \supset \ldots \supset \sim\alpha_{i_s} \supset \alpha \supset \gamma) \supset \alpha \supset (\beta \supset \gamma)$ [VII, 3, 4, 5]
10. $\sim(\sim\delta \supset \sim\alpha_{i_1} \supset \ldots \supset \sim\alpha_{i_s} \supset \alpha \supset \gamma) \supset \beta \supset \gamma$ [II, 9, 6]
11. $\sim(\sim\delta \supset \sim\alpha_{i_1} \supset \ldots \supset \sim\alpha_{i_s} \supset \alpha \supset \gamma) \supset \beta$ [II, 8, 6]
12. $\sim(\sim\delta \supset \sim\alpha_{i_1} \supset \ldots \supset \sim\alpha_{i_s} \supset \alpha \supset \gamma) \supset \gamma$ [II, 10, 11)]

$\sim\delta \supset \sim\alpha_{i_1} \supset \ldots \supset \sim\alpha_{i_s} \supset \alpha \supset \gamma$ [III, 12, 7];

and thus $\alpha \supset \gamma \epsilon \mathrm{A}$.

L4. *If* $\sim\alpha \supset \beta \epsilon \mathrm{A}$, $\sim\alpha \supset \sim\beta \epsilon \mathrm{A}$, *then* $\alpha \epsilon \mathrm{A}$.

Proof. Similar to that of L3.

L5. $\sim(\alpha \supset \beta) \supset \alpha \epsilon \mathrm{A}$.

Proof. Consider any A_n, as before.

1. $\sim(\sim\delta \supset \sim\alpha_{i_1} \supset \ldots \supset \sim\alpha_{i_s} \supset \sim(\alpha \supset \beta) \supset \alpha) \supset \sim(\alpha \supset \beta)$ [VIa]
2. $\sim(\sim\delta \supset \sim\alpha_{i_1} \supset \ldots \supset \sim\alpha_{i_s} \supset \sim(\alpha \supset \beta) \supset \alpha) \supset \sim\alpha$ [VIb]
3. $\sim(\sim\delta \supset \sim\alpha_{i_1} \supset \ldots \supset \sim\alpha_{i_s} \supset \sim(\alpha \supset \beta) \supset \alpha) \supset \alpha$ [1, IV, I]

$\sim\delta \supset \sim\alpha_{i_1} \supset \ldots \supset \sim\alpha_{i_s} \supset \sim(\alpha \supset \beta) \supset \alpha$ [III, 3, 2],

and $\sim(\alpha \supset \beta) \supset \alpha \epsilon \mathrm{A}$.

L6. $\sim(\alpha \supset \beta) \supset \sim\beta \epsilon \mathrm{A}$.

Proof. Similar to that of L5.

L7. $\mathrm{Th} \epsilon \mathrm{A}$ [L2—L6]

L8. A *is consistent* (*in the classical sense*).

Proof. Had A_0 been inconsistent, then there would be α such that $\sim\delta \supset \alpha$ and $\sim\delta \supset \sim\alpha$. Then by III δ would have to be a theorem, contrary to the supposition that it is not. A_0 must therefore be consistent. Suppose that A_n is the lowest inconsistent class, so that A_{n-1} is consistent. Then

1. $\sim\delta \supset \sim\alpha_{i_1} \supset \ldots \supset \sim\alpha_{i_s} \supset \beta$ where α_{i_s} is α_n.
2. $\sim\delta \supset \sim\alpha_{i_1} \supset \ldots \supset \sim\alpha_{i_s} \supset \sim\beta$
3. $\sim(\sim\delta \supset \sim\alpha_{i_1} \supset \ldots \supset \alpha_{i_s}) \supset \sim\delta$ [VIa]
4. $\sim(\sim\delta \supset \sim\alpha_{i_1} \supset \ldots \supset \alpha_{i_s}) \supset \sim\alpha_{i_j}$ for each j, $1 \leqq j \leqq s$ [VI]
5. $\sim(\sim\delta \supset \sim\alpha_{i_1} \supset \ldots \supset \alpha_{i_s}) \supset \beta$ [VII, 3, 4, 1]
6. $\sim(\sim\delta \supset \sim\alpha_{i_1} \supset \ldots \supset \alpha_{i_s}) \supset \sim\beta$ [VII, 3, 4, 2]
7. $\sim\delta \supset \sim\alpha_{i_1} \supset \ldots \supset \alpha_{i_s}$ [III, 5, 6]
8. $\alpha_n \epsilon A_{n-1}$ [7]

   $A_n = A_{n-1}$ [8],

and A_{n-1} is also inconsistent, contrary to the supposition.

L9. $\delta \tilde{\epsilon} A$

Proof. 1. $\sim\delta \supset \sim\delta$ [IX]

2. $\sim\delta \epsilon A_0$ [1]
3. $\sim\delta \epsilon A$ [2]

   $\delta \tilde{\epsilon} A$ [L8, 3]

L10. *If* $\alpha \tilde{\epsilon} A$, *then* $\sim\alpha \epsilon A$.

Proof. α, as a formula, appears somewhere in the sequence B, say $\alpha = \alpha_{n+1}$. If $\alpha \tilde{\epsilon} A$, then $\alpha \tilde{\epsilon} A_n$. In case $\alpha_{n+1} \tilde{\epsilon} A_n$ the definition of A_{n+1} asserts that $A_{n+1} = \hat{\alpha}\ (\sim\alpha_{n+1} \supset \alpha \epsilon A_n)$. If $A_n = \hat{\alpha}(\sim\delta \supset \sim\alpha_{i_1} \supset \ldots \supset \sim\alpha_{i_s} \supset \alpha \epsilon \mathrm{Th})$ then $A_{n+1} = \hat{\alpha}(\sim\delta \supset \sim\alpha_{i_1} \supset \ldots \supset \sim\alpha_{i_s} \supset \sim\alpha_{n+1} \supset \alpha \epsilon \mathrm{Th})$. From XIV we have that $\sim\delta \supset \sim\alpha_{i_1} \supset \ldots \supset \sim\alpha_{i_s} \supset \sim\alpha_{n+1} \supset \sim\alpha_{n+1} \epsilon \mathrm{Th}$. Therefore $\sim\alpha_{n+1} \epsilon A_{n+1}$ and $\sim\alpha \epsilon A$.

L11. *If* $\alpha \supset \beta \epsilon A$ *and* $\alpha \epsilon A$, *then* $\beta \epsilon A$.

Proof. 1. $\sim\delta \supset \sim\alpha_{i_1} \supset \ldots \supset \sim\alpha_{i_s} \supset \alpha \supset \beta$

2. $\sim\delta \supset \sim\alpha_{i_1} \supset \ldots \supset \sim\alpha_{i_s} \supset \alpha$ for some sufficiently large i_s.
3. $\sim(\sim\delta \supset \sim\alpha_{i_1} \supset \ldots \supset \sim\alpha_{i_s} \supset \beta) \supset \sim\delta$ [VIa]
4. $\sim(\sim\delta \supset \sim\alpha_{i_1} \supset \ldots \supset \sim\alpha_{i_s} \supset \beta) \supset \sim\alpha_{i_j}$ for each j, $1 \leqq j \leqq s$ [VIa]
5. $\sim(\sim\delta \supset \sim\alpha_{i_1} \supset \ldots \supset \sim\alpha_{i_s} \supset \beta) \supset \sim\beta$ [VIb]
6. $\sim(\sim\delta \supset \sim\alpha_{i_1} \supset \ldots \supset \sim\alpha_{i_s} \supset \beta) \supset \alpha \supset \beta$ [VII, 3, 4, 1]
7. $\sim(\sim\delta \supset \sim\alpha_{i_1} \supset \ldots \supset \sim\alpha_{i_s} \supset \beta) \supset \alpha$ [VII, 3, 4, 2]
8. $\sim(\sim\delta \supset \sim\alpha_{i_1} \supset \ldots \supset \sim\alpha_{i_s} \supset \beta) \supset \beta$ [II, 6, 7]

   $\beta \epsilon A$ [III, 5, 8]

L12. *If* $\beta\epsilon$A, *then* $\alpha \supset \beta\epsilon$A

Proof. 1. $\beta \supset (\alpha \supset \beta)\epsilon$Th [XII]

2. $\beta \supset (\alpha \supset \beta)\epsilon$A [L7, 1]

$\alpha \supset \beta\epsilon$A [L11, 2]

L13. *If* $\alpha\bar{\epsilon}$A, *then* $\alpha \supset \beta\epsilon$A

Proof. 1. $\alpha\bar{\epsilon}$A

2. $\sim\alpha\epsilon$A [L10, 1]

3. $\sim\alpha \supset (\alpha \supset \beta)\epsilon$A [XIII, L7]

$\alpha \supset \beta$ [L11, 2, 3]

L14. *If* $\alpha\epsilon$A *and* $\beta\bar{\epsilon}$A, *then* $\alpha \supset \beta\bar{\epsilon}$A [L11]

L15a. *If* $\alpha\epsilon$A, *then* $\sim\alpha\bar{\epsilon}$A [L8]

L15b. *If* $\alpha\bar{\epsilon}$A, *then* $\sim\alpha\epsilon$A.

Proof. Suppose that α is α_{n+1} and

1. $\alpha\bar{\epsilon}$A

2. $A_{n+1} = \hat{\alpha}(\sim\delta \supset \sim\alpha_i \supset \ldots \supset \sim\alpha_{n+1} \supset \alpha)$ [1]

3. $\sim\delta \supset \sim\alpha_{i_1} \supset \ldots \supset \sim\alpha_{n+1} \supset \sim\alpha_{n+1}$ [XIV]

4. $\sim\alpha_{n+1}\epsilon A_{n+1}$ [3]

$\sim\alpha_{n+1}\epsilon$A [4].

THE COMPLETENESS THEOREM. *If* $\delta\bar{\epsilon}$Th, *then* δ *is not satisfied by* M_2. *Proof.* If $\delta\bar{\epsilon}$Th, then there is a class of formulas A (depending on δ) such that $\delta\bar{\epsilon}$A (by L9), and such that were M_2 to satisfy δ, then (by L12—L15) δ would belong to A. Thus δ is not satisfied by M_2.

4. Independence. Consider the following matrices:

MI.

$\supset$	1	2	3	4	$\sim$
*1	1	1	4	4	4
2	1	1	4	4	3
3	2	2	1	1	2
4	1	1	1	1	1

MII.

$\supset$	1	2	3	4	$\sim$
*1	1	2	3	3	4
2	1	1	3	3	3
3	1	2	1	2	2
4	1	1	1	1	1

MIII.

$\supset$	1	2	$\sim$
*1	1	1	2
2	1	1	1

MIV.

$\supset$	1	2	3	$\sim$
*1	1	3	3	3
2	1	1	3	1
3	1	1	1	1

MV.

$\supset$	1	2	3	$\sim$
*1	1	3	3	3
*2	1	1	2	3
3	1	1	1	1

For completeness of the system rule I is necessary because M I satisfies IV and V, and II and III are valid for M I but I is not valid for M I (take $\alpha = 3$, $\beta = 4$, $\gamma = 2$). I is not derivable from the remaining rules. But, $\sim\alpha \supset (\alpha \supset \beta)$ is not satisfied by M I (take $\alpha = 2$, $\beta = 1$) and is thus unprovable without I.

Similarly, without II the system would not be complete. For, M II satisfies IV and V, and I and III are valid for M II but II is not valid for M II (take $\alpha = 3$, $\beta = 1$, $\gamma = 4$) and $(\alpha \supset (\beta \supset \gamma)) \supset ((\alpha \supset \beta) \supset (\alpha \supset \gamma))$ is not satisfied (same substitution).

$\sim((\alpha \supset \alpha) \supset \sim(\alpha \supset \alpha))$ is not satisfied by M III and III is not valid for M III (take $\alpha = 2$). IV and V are satisfied by M III and the rules I and II valid for it. Thus, III is necessary for completeness.

The rules I, II, and III are valid for M IV, and V is satisfied by M IV. But, IV is not (take $\alpha = 2$, $\beta = 3$). Hence, IV is independent.

The rules I, II, and III are valid for M V and IV is satisfied by M V; however V is not (take $\alpha = 1$, $\beta = 2$). Therefore, V is independent.[8]

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