Completeness in Propositional *vs* Predicate Logic

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In Section 2 of Hunter, we examined a Henkin-type proof of the completeness of the formal system PS of propositional logic. In section 3, completeness of the system QS of predicate logic is also proved in the style of Henkin. Although the basic strategy of the two proofs is the same, there are several significant differences, which is what this handout is about.

The following table summarizes the heart of the proof for PS — 32.13: Every p–consistent set Γ of wffs of P has a model.

Proof Theory		**Model Theory**
An arbitrary p-consistent set Γ of P		Γ has a model.
$\cap$ (is *a subset of*: 32.12)		$\Uparrow$
A maximal p-consistent set Γ' of PS	$\overset{32.13}{\Longrightarrow}$	Γ' has a model

There is a great deal more complexity in the language Q than in the language P, the semantics for Q *versus* the semantics for P, and in the formal systems QS *vs* PS. This complexity leads to some different notions in the completeness proof, the heart of which is — 45.14: Every consistent, negation-complete, closed first order theory (fot) K′ has a denumerable model.

Proof Theory		**Model Theory**
An arbitrary consistent fot K		K has a denumerable model.
$\Cap$ (is *extended by*: 45.13)		$\Uparrow$
A consistent, negation-complete, closed fot K′	$\overset{45.14}{\Longrightarrow}$	K′ has a denumerable model

A trivial difference is that we speak of "consistent" sets of formulas of Q (or Q^+) rather than "p-consistent" sets of formulas of P. The reason for this is that there is no notion in the semantics for Q corresponding to that of m-consistency for P. In the semantics for P, if a set has no model, then for all interpretations, there is at least one formula in the set that is *false*. But in the semantics for Q, some sets have no models because some of the formulas in the set are simply *not true* in the sense that they *lack truth values* (truth and falsehood are defined in terms of being satisfied by *all* or *no* I-sequences). We might have formulas in the set which contain free variables, and which are satisfied by some I-sequences but not others. We do not want to say that such sets are *inconsistent*, since they have some "positive semantic value" (so to speak). Specifically, they are *simultaneously satisfiable*. The correlate of an m-inconsistent set would be a set that is *not simultenously satisfiable*.

A second difference is that in the predicate logic proof, we do not reason about arbitrary sets of QS, but rather about *first order theories* (fots). Such theories have a determinate (and special) structure, which aids in carrying out the proof of 45.14.

A third difference follows from the second. The relevant relation between theories is not that of superset ($\Gamma \subset \Gamma'$), but one of *extension* ($K \Subset K'$). A formal system S′ is an extension of a formal system S if and only if all the theorems of S are theorems of S′. Note that a first order theory *is* a formal system, so the definition of 'extension' applies to theories.

A fourth difference is that rather than reasoning about maximal p-consistent sets of PS, we reason about negation-complete consistent first order theories. Producing a first order theory that is negation-complete is quite similar to producing a maximal p-consistent set. This is why the Lindenbaum constructions for the two cases (32.12 and 45.10) are so similar.

A fifth difference is that there is a property of a first order theory, that of being *closed*, which has no analogue in propositional logic. It will turn out that this property is absolutely crucial in the proof of 45.14.

The final difference is on the semantical side of the table. Since interpretations of Q (or Q^+) have a domain, the size of the domain may be relevant to whether it is a model. A denumerable model is a a model the members of whose domain stand in a 1-to-1 relation to the natural numbers. We now represent the two proofs of completeness in a single table.

PS	QS
32.12 Every p-consistent set of PS is a subset of some maximal p-consistent set of PS.	45.10 If K is a consistent first-order theory, then there is a fot K′ that is a consistent negation-complete extension of K with the same formulas as K.
	45.13 If K is a consistent first-order theory, then there is a *closed* fot K′ that is a consistent negation-complete extension of K with the same formulas as K.
	45.14 Every closed fot that is consistent and negation-complete has a denumerable model.
32.13 Every p-consistent set of PS has a model.	45.15 Every consistent fot has a deumerable model.

Obviously there is a big gap between the relevant metatheorems in predicate logic that does not exist in propositional logic. The reason is that dealing semantically with free variables in a formula is a complicated affair — one which is perhaps best avoided! The extra work involved is designed to allow us to work with systems of predicate logic in which the presence of free variables in formulas is of no consequence, *i.e.*, in *closed* systems. Here is the definition of a *closed* fot:

> **Definition.** Let K be a first order theory with at least one closed term. K is said to be a *closed* first order theory iff every formula A containing exactly one free variable v is such that *if* $\vdash_K Ac/v$ *for all closed* c, *then* $\vdash_K \bigwedge vA$.

For example, consider the formula $F^{*\prime}x'x'$ of a fot K. If K is closed, then *if* $\vdash_K F^{*\prime}cc$ for *all* closed terms c of Q, *then* $\vdash_K \bigwedge x'F^{*\prime}x'x'$. More abstractly, given that v (and only v) occurs free in A, K's being a closed fot implies that *if* $\vdash_K Ac/v$ for *every* closed term c of K, *then* $\vdash_K \bigwedge vA$. We can ensure that this property holds when we build extensions of first order theories by adding, for each closed term c, formulas of the form $Ac/v \supset \bigwedge vA$ as *proper axioms.* This will catch all the formulas Ac/v which are theorems of the first order theory and (by application of Modus Ponens) will guarantee that the quantified formula $\bigwedge vA$ is a theorem *if* Ac/v is a theorem for *every* closed term c of K. This is what 45.13 does.

Specifically, we take our given consistent first order theory K, and we *extend* it in the following ways:

1. Form K_0 by adding denumerably many new constant symbols with an effective enumeration $\langle b_1, b_2, \ldots\rangle$ to K. And, let $\langle A_1, A_2, \ldots\rangle$ be an enumeration of the wffs of K_0 that contain only one free variable.
2. Form K_1 by adding S_1 as a proper axiom to K_0, where:

   $$S_1 \stackrel{\text{def}}{=} A_1 b_{j_1}/v_1 \supset \bigwedge v_1 A_1$$

   Here, b_{j_1} is the first of the b_i's that does not occur in A_1, and v_1 is the (only) variable occurring free in A_1.
3. Form K_2 by adding S_2 as a proper axiom to K_1, where:

   $$S_2 \stackrel{\text{def}}{=} A_2 b_{j_2}/v_2 \supset \bigwedge v_2 A_2$$

   $$\vdots$$

n. Form K_{n-1} by adding S_{n-1} as a proper axiom to K_{n-2}. Repeat this construction for *all* $n > 3$.

Finally, form K_∞, by adding all of the denumerably many S_i's to K_0 as proper axioms. Hunter's proof of 45.13 shows (in a very straightforward way) that K_∞ is a *consistent, closed* extension of K. And, by 45.10, we know how to extend K_∞ to K', which is *consistent* and *negation-complete.* It is (again) straightforward to show that K' is also a *closed* first-order theory. [This procedure allows us to dispense with any reference to I-sequences in our metaproofs. In the semantics for Q, a universally quantified formula is satisfied by an I-sequence s if and only if the formula without the quantifier is satisfied by all I-sequences which differ at most from s in the assignment to the variable following the quantifier. Working with *closed* theories allows us to *syntactically simulate* this semantical property of "satisfaction by an I-sequence", *substitutionally.*]

Once we have 45.13 proved, we just need to find a *denumerable model* for our consistent, negation-complete, and closed extension K' of K. This involves two clever (Henkin) tricks, which are described in detail in my handout on 45.14. The first trick is to make the domain of our Henkin interpretation of K' the set of closed terms of K' itself.

There is one more trick that needs to be played in order to complete the argument that closes the gap between 45.10 and 45.14. Recall that in the completeness proof for propositional logic, we specified that our interpretation would assign truth to a formula just in case it was a member of the maximal consistent set (Γ') under consideration. In predicate logic, we use a similar technique. One difference is that we assign the value *true* to just those formulas which are *theorems* of our extended first order theory (K'). The other is that we assign as extensions to predicates (formally, for n-place predicates, sets of n-tuples of members of the domain) just those strings of n terms which follow the predicates *in theorems of* the system (K'). This will guarantee that all closed atomic formulas which are theorems come out true in the interpretation.

The rest is just mopping up. It is easy to show that given this highly specialized, highly contrived (Henkin) interpretation, all the theorems of a consistent, closed, negation-complete first order theory come out true [45.14]. So that theory has a model, and if it does, the theory of which it is an extension (K) has a model, which is the ultimate result we have been seeking.

A final word about the comparison between this completeness "proof" and the completeness "proof" for propositional logic. We were able to complete the main crux of the argument for completeness [45.10–45.14] *without using* the fine-grained semantical concept "satisfaction by an I-sequence". But, there is a result, 45.17, which does *not* do an end-run around the central semantical category of satisfaction by an I-sequence. In fact, it is very much like the result for propositional logic.

PS	QS
32.13 Every p-consistent set of PS has a model.	45.17 Every consistent set of formulas of a first order theory is simultaneously satisfiable in a deumerable model.

This result (as proved in Hunter) is a consequence of 45.16, which in turn is a consequence of 45.15. So we get a result about satisfiablity from results that make the semantical role of satisfiability moot, which is a rather curious thing.