A More Straightforward Proof of Metatheorem 40.12

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40.12: *Let I be an arbitrary interpretation with domain D. Let A be an arbitrary wff. Let s and s' be two sequences such that, for each free variable v in A, if v is the kth variable in the fixed enumeration of the variables of Q, then s and s' have the same member of D for their kth terms. Then s satisfies A iff s' does.*

Proof. By induction on the complexity of A, *i.e.*, on the number n of connectives and quantifiers in A.

Basis Step: $n = 0$. There are two possible cases.

1. A is a propositional symbol p.
This is trivial. Given any interpretation I and a propositional symbol p either every sequence satisfies p or every sequence fails to satisfy p.

2. A is of the form $Ft_1, \ldots t_n$.
We know that s satisfies A iff $\langle t_1 \star s, \ldots, t_n \star s \rangle \in I(F)$ and s' satisfies A iff $\langle t_1 \star s', \ldots, t_n \star s' \rangle \in I(F)$. We prove that for each t_i in A, that $t_i \star s = t_i \star s'$, and therefore that s satisfies A iff s' satisfies A. The proof proceeds by induction on the number m of function symbols in t_i.

> Basis Step: $m = 0$. There are two sub-cases.
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> (i) t_i is a constant. In this case, $t_i \star s = t_i \star s' = I(t_i)$.
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> (ii) t_i is a variable v_k. In this case, since A is *atomic*, t_i is free in A. By the hypothesis of the theorem, the kth term of s = the kth term of s'. Thus, $t_i \star s = t_i \star s'$.
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> Inductive Step:
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> (iii) t_i is f$t_{j_1}, \ldots, t_{j_m}$, which contains a *total* of $m > 0$ function symbols (including the leading f)
>
> Let I(f) $= f$. Then, $t_i \star s = ft_{j_1} \star s, \ldots, t_{j_m} \star s$, and $t_i \star s' = ft_{j_1} \star s', \ldots, t_{j_m} \star s'$. By the inductive hypothesis, we know that for all i such that $1 \leq i \leq m$, $t_{j_i} \star s = t_{j_i} \star s'$. This is because the t_{j_i} have $< m$ function symbols in them (t_i has m function symbols, including the leading f). Thus $t_i \star s = t_i \star s'$, since identical functions have the same values given identical arguments.

Inductive Step: There are three cases.

1. $A = {\sim}B$. Trivial. By the inductive hypothesis, s satisfies B iff s' satisfies B. Therefore, s does not satisfy B iff s' does not satisfy B. So, s satisfies A iff s' satisfies A.

2. $A = B \supset C$. Once again trivial. Simple exercise.

3. $A = \bigwedge_{v_p} B$. Assume that s satisfies A. Then, $s(d_i/p)$ satisfies B for each d_i in D. We will show that $s'(d_i/p)$ satisfies B for each d_i in D. To show this, we first show that for each d_i, $s(d_i/p)$ and $s'(d_i/p)$ have the same kth term for each free variable v_k in B. First note that every variable that is free in A is also free in B (B has one fewer quantifier in it than A, but is otherwise identical to A). By the hypothesis of the theorem, for each free v_j in A, s and s' have the same jth term. For each such v_j, s and $s(d_i/p)$ have the same jth term, as do s' and $s'(d_i/p)$, (since s and $s(d_i/p)$ differ *at most* in the pth term and for all j such that v_j is free in A, $j \neq p$, since v_p is not free in A). If, then, v_k is free in A, $s(d_i/p)$ and $s'(d_i/p)$ have the same kth term. The only other variable that could be free in B is v_p. Since the pth term of $s(d_i/p)$ and $s'(d_i/p)$ is d_i, we have shown that for every free variable v_k in B, $s(d_i/p)$ and $s'(d_i/p)$ have the same kth term. We can, therefore, employ the inductive hypothesis. We have, then, that for each d_i, $s(d_i/p)$ satisfies B iff $s'(d_i/p)$ satisfies B. It follows, then, that for each d_i, $s'(d_i/p)$ satisfies B. Therefore, s' satisfies A. The other direction proceeds in exactly the same manner. It is left as a simple exercise. □