Natural Deduction Rules for the In-Class Final

Philosophy 12A

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Rule of Assumptions:

j	(j)	p	Assumption (or: Premise)

where p may be any formula (or a premise of the sequent).

Rule of &-Elimination (&E):

$a_1, \ldots, a_n$	(j)	$p \,\&\, q$	
	⋮		
$a_1, \ldots, a_n$	(k)	p	j &E
	or		
$a_1, \ldots, a_n$	(k)	q	j &E

Rule of &-Introduction (&I):

$a_1, \ldots, a_n$	(j)	p	
	⋮		
$b_1, \ldots, b_u$	(k)	q	
	⋮		
$a_1, \ldots, a_n, b_1, \ldots, b_u$	(m)	$p \,\&\, q$	j, k &I

Rule of →-Elimination (→E):

$a_1, \ldots, a_n$	(j)	$p \to q$	
	⋮		
$b_1, \ldots, b_u$	(k)	p	
	⋮		
$a_1, \ldots, a_n, b_1, \ldots, b_u$	(m)	q	j, k →E

Rule of →-Introduction (→I):

j	(j)	p	Assumption
	⋮		
$a_1, \ldots, a_n$	(k)	q	
	⋮		
$\{a_1, \ldots, a_n\}/j$	(m)	$p \to q$	j, k →I

Rule of ∼-Elimination (∼E):

$a_1, \ldots, a_n$	(j)	$\sim q$	
	⋮		
$b_1, \ldots, b_u$	(k)	q	
	⋮		
$a_1, \ldots, a_n, b_1, \ldots, b_u$	(m)	⅄	j, k ∼E

Rule of ∼-Introduction (∼I):

j	(j)	p	Assumption
	⋮		
$a_1, \ldots, a_n$	(k)	⅄	
	⋮		
$\{a_1, \ldots, a_n\}/j$	(m)	$\sim p$	j, k ∼I

Rule of Double Negation (DN):

$a_1, \ldots, a_n$	(j)	$\sim\sim p$	
	⋮		
$a_1, \ldots, a_n$	(k)	p	j DN

Rule of ∨-Introduction (∨I):

$a_1, \ldots, a_n$	(j)	p	
	⋮		
$a_1, \ldots, a_n$	(k)	$p \vee q$	j ∨I
	or		
$a_1, \ldots, a_n$	(k)	$q \vee p$	j ∨I

Rule of ∨-Elimination (∨E):

$a_1, \ldots, a_n$	(g)	$p \vee q$	
	⋮		
h	(h)	p	Assumption
	⋮		
$b_1, \ldots, b_u$	(i)	r	
	⋮		
j	(j)	q	Assumption
	⋮		
$c_1, \ldots, c_w$	(k)	r	
	⋮		
$\mathcal{X}$	(m)	r	g, h, i, j, k ∨E

$\mathcal{X} = \{a_1, \ldots, a_n\} \cup \{b_1, \ldots, b_u\}/h \cup \{c_1, \ldots, c_w\}/j$

Rule of Definition for ↔ (Df):

$a_1,\ldots, a_n$	(j)	$(p \rightarrow q) \,\&\, (q \rightarrow p)$	
	$\vdots$		
$a_1,\ldots, a_n$	(k)	$p \leftrightarrow q$	j Df

OR

$a_1,\ldots, a_n$	(j)	$p \leftrightarrow q$	
	$\vdots$		
$a_1,\ldots, a_n$	(k)	$(p \rightarrow q) \,\&\, (q \rightarrow p)$	j Df

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Rule of ∃-Introduction:

$a_1,\ldots, a_n$	(j)	$\phi\tau$	
	$\vdots$		
$a_1,\ldots, a_n$	(k)	$(\exists\nu)\phi\nu$	j ∃I

$\ulcorner(\exists\nu)\phi\nu\urcorner$ is obtained from $\phi\tau$ by replacing *one or more* occurrences of τ in $\phi\tau$ by an individual variable ν (*which must not occur in* $\phi\tau$) and then prefixing the quantifier $\ulcorner(\exists\nu)\urcorner$.

Rule of ∀-Elimination:

$a_1,\ldots, a_n$	(j)	$(\forall\nu)\phi\nu$	
	$\vdots$		
$a_1,\ldots, a_n$	(k)	$\phi\tau$	j ∀E

$\phi\tau$ is obtained from $\ulcorner(\forall\nu)\phi\nu\urcorner$ by deleting the quantifier prefix $\ulcorner(\forall\nu)\urcorner$ and then replacing *every* occurrence of ν in the open sentence $\phi\nu$ by *one and the same* constant τ.

Rule of ∀-Introduction:

$a_1,\ldots, a_n$	(j)	$\phi\tau$	
	$\vdots$		
$a_1,\ldots, a_n$	(k)	$(\forall\nu)\phi\nu$	j ∀I

Where τ is not in any of the formulae on lines $a_1,\ldots, a_n$ and ν is not in $\phi\tau$. $\ulcorner(\forall\nu)\phi\nu\urcorner$ is obtained by replacing *every* occurrence of the constant τ in $\phi\tau$ with the variable ν and then prefixing the universal quantifier $\ulcorner(\forall\nu)\urcorner$.

Rule of ∃-Elimination:

$a_1,\ldots, a_n$	(i)	$(\exists\nu)\phi\nu$	
	$\vdots$		
j	(j)	$\phi\tau$	Assumption
	$\vdots$		
$b_1,\ldots, b_u$	(k)	$\mathscr{P}$	
	$\vdots$		
$\mathcal{X}$	(m)	$\mathscr{P}$	i, j, k ∃E

Where τ is not in (*i*) $\ulcorner(\exists\nu)\phi\nu\urcorner$, (*ii*) $\mathscr{P}$, *or* (*iii*) *any of* the formulae $b_1,\ldots, b_u$ (other than $\phi\tau$ itself). The set of line numbers at line m is: $\mathcal{X} = \{a_1,\ldots, a_n\} \cup \{b_1,\ldots, b_u\}/j$.

Rule of Sequent Introduction (SI):

Suppose the sequent $r_1,\ldots,r_n \vdash s$ is a *substitution-instance* of a sequent $p_1,\ldots,p_n \vdash q$ which appears in our Official List of Sequents and Theorems, and that the formulae $r_1,\ldots,r_n$ occur at lines $j_1,\ldots,j_n$ in a proof. Then we may infer s at line k, labeling the line '$j_1,\ldots,j_n$ SI (Identifier)' and writing on the left all the numbers which appear on the left of lines $j_1,\ldots,j_n$. The "Identifier" for a sequent is given by its name in our Official List of Sequents and Theorems (see below).

Rule of Theorem Introduction (TI):

If $\vdash s$ is a *substitution-instance* of some theorem $\vdash q$ which appears in our Official List of Sequents and Theorems, we may introduce a new line k into a proof with the formula s at it and no numbers on its left, labeling the line 'TI (Identifier)'. The *only* theorem in our Official List is LEM — see below.

Our Official List of Sequents and Theorems

$A \vee B, {\sim}A \vdash B$; or; $A \vee B, {\sim}B \vdash A$	(DS)
$A \rightarrow B, {\sim}B \vdash {\sim}A$	(MT)
$A \vdash B \rightarrow A$	(PMI)
${\sim}A \vdash A \rightarrow B$	(PMI)
$A \vdash {\sim}{\sim}A$	(DN$^+$)
${\sim}(A \,\&\, B) \dashv\vdash {\sim}A \vee {\sim}B$	(DeM)
${\sim}(A \vee B) \dashv\vdash {\sim}A \,\&\, {\sim}B$	(DeM)
${\sim}({\sim}A \vee {\sim}B) \dashv\vdash A \,\&\, B$	(DeM)
${\sim}({\sim}A \,\&\, {\sim}B) \dashv\vdash A \vee B$	(DeM)
$A \rightarrow B \dashv\vdash {\sim}A \vee B$	(Imp)
${\sim}(A \rightarrow B) \dashv\vdash A \,\&\, {\sim}B$	(Neg-Imp)
$A \,\&\, (B \vee C) \dashv\vdash (A \,\&\, B) \vee (A \,\&\, C)$	(Dist)
$A \vee (B \,\&\, C) \dashv\vdash (A \vee B) \,\&\, (A \vee C)$	(Dist)
⅄ $\vdash A$	(EFQ)
$A * B \vdash B * A$	(Com)
${\sim}{\sim}A * {\sim}{\sim}B \dashv\vdash A * B$	(SDN)
$A * B \dashv\vdash {\sim}{\sim}A * B \dashv\vdash A * {\sim}{\sim}B$	(SDN)
$\vdash A \vee {\sim}A$	(LEM)
$(\forall\nu){\sim}\phi\nu \dashv\vdash {\sim}(\exists\nu)\phi\nu$	(QS)
$(\exists\nu){\sim}\phi\nu \dashv\vdash {\sim}(\forall\nu)\phi\nu$	(QS)
$\phi\nu \dashv\vdash \phi\nu'$	(AV)

Notes on Our Official List of Sequents and Theorems.

In (Com), '$*$' can be any binary LSL connective *except* '→'. In (SDN), '$*$' can be *any* binary LSL connective. In (AV), $\phi\nu$ must be *closed*, and $\phi\nu'$ must be an *alphabetic variant* of $\phi\nu$. For example, '$(\forall x)Fx$' and '$(\forall y)Fy$' are both *closed* sentences (*i.e.*, they have *no free variables* in them), and they are *alphabetic variants* of each other, because they differ *only* in which individual variable ('x' or 'y') is used, but they have exactly the same *logical* (*i.e.*, *syntactical*) *structure*.