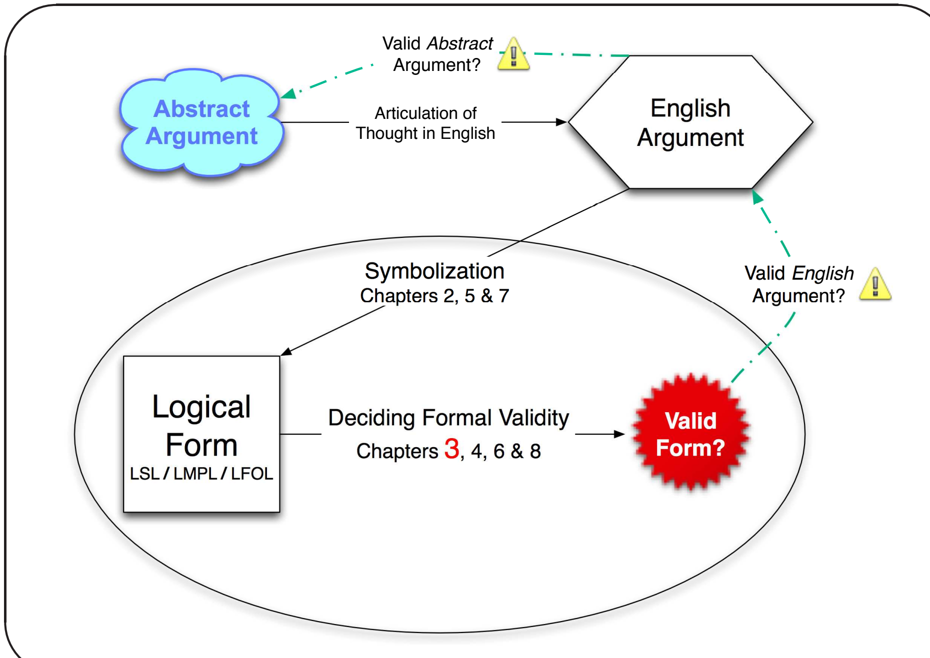
Announcements & Such

- *Leonard Cohen*
- Administrative Stuff
  - **HW #2 1st-submissions are due Today (4pm, drop box).**
    - ∗ Note: This involves problems from chapters 2 and 3.
    - ∗ Consult the *HW Tips Handout* for helpful tips on HW #2.
  - ☞ **Homework formatting. Please put the following information:**
    - ∗ **Name, GSI, section time, and date.**

    on all assignments and exams (upper-right corner of first page).
- Chapter 3 — *Truth-Functional Semantics* for LSL
  - The truth-functions and the LSL connectives
  - Truth-Tables — a tool for "seeing" LSL's "logically possible worlds"
  - *Formal* explications of Logical truth, validity, *etc.* — in LSL



Chapter 3 — Semantics of LSL: Truth Functions I

- The semantics of LSL is *truth-functional* — the truth value of a compound statement is a function of the truth values of its parts.
- Truth-conditions for each of the five LSL statement forms are given by *truth tables*, which show how the truth value of each type of complex sentence depends on the truth values of its constituent parts.
- Truth-tables provide a very precise way of thinking about *logical possibility*. Each row of a truth-table can be thought of as a *way the world might be*. The actual world falls into *exactly one* of these rows.
- In this sense, truth-tables provide a way to "see" "logical space."
- Truth-tables will also provide us with a rigorous way to establish whether an argument form in LSL is valid (*i.e.*, sentential validity).
- We just look for rows of a salient truth-table in which all the premises are true and the conclusion is false. That's where we're headed.

Chapter 3 — Semantics of LSL: Truth Functions II

- We begin with negations, which have the simplest truth functions. The truth table for negation is as follows (we use $\top$ and $\bot$ for true and false):

p	$\sim p$
$\top$	$\bot$
$\bot$	$\top$

- In words, this table says that if p is true than $\sim p$ is false, and if p is false, then $\sim p$ is true. This is quite intuitive, and corresponds well to the English meaning of 'not'. Thus, LSL negation is like English negation.
- Examples:
  - It is not the case that Wagner wrote operas. ($\sim W$)
  - It is not the case that Picasso wrote operas. ($\sim P$)
- '$\sim W$' is false, since 'W' is true, and '$\sim P$' is true, since 'P' is false (like English).

Chapter 3 — Semantics of LSL: Truth Functions III

p	q	$p \,\&\, q$
⊤	⊤	⊤
⊤	⊥	⊥
⊥	⊤	⊥
⊥	⊥	⊥

- Notice how we have four (4) rows in our truth table this time (not 2), since there are four possible ways of assigning truth values to p and q.
- The truth-functional definition of & is very close to the English 'and'. A LSL conjunction is true if *both* conjuncts are true; it's false otherwise.
  - Monet and van Gogh were painters. ($M \,\&\, V$)
  - Monet and Beethoven were painters. ($M \,\&\, B$)
  - Beethoven and Einstein were painters. ($B \,\&\, E$)
- '$M \,\&\, V$' is true, since both 'M' and 'V' are true. '$M \,\&\, B$' is false, since 'B' is false. And, '$B \,\&\, E$' is false, since 'B' and 'E' are both false (like English).

Chapter 3 — Semantics of LSL: Truth Functions IV

p	q	$p \vee q$
⊤	⊤	⊤
⊤	⊥	⊤
⊥	⊤	⊤
⊥	⊥	⊥

- Our truth-functional ∨ is not as close to the English 'or'. An LSL disjunction is true if *at least one* disjunct is true (false otherwise).
- In English, 'A or B' often implies that 'A' and 'B' are *not both true*. That is called *exclusive* or. In LSL, '$A \vee B$' is *not* exclusive; it is *inclusive* (true if both disjuncts are true). But, we *can* express exclusive or in LSL. How?
  - Either Jane austen or René Descartes was novelist. ($J \vee R$)
  - Either Jane Austen or Charlotte Bronte was a novelist. ($J \vee C$)
  - Either René Descartes or David Hume was a novelist. ($R \vee D$)
- The first two disjunctions are true because at least one their disjuncts is true, but the third is false, since both of its disjuncts are false.

Chapter 3 — Semantics of LSL: Truth Functions V

p	q	$p \to q$
⊤	⊤	⊤
⊤	⊥	⊥
⊥	⊤	⊤
⊥	⊥	⊤

- Our truth-functional → is farther from the English 'only if'. An LSL conditional is false iff its antecedent is true and its consequent is false.
- Consider the following English conditionals. [M = 'the moon is made of green cheese', O = 'life exists on other planets', and E = 'life exists on Earth']
  - If the moon is made of green cheese, then life exists on other planets.
  - If life exists on other planets, then life exists on earth.
- The LSL translations of these sentences are both true. '$M \to O$' is true because its antecedent 'M' is false. '$O \to E$' is true because its consequent 'E' is true. This seems to deviate from the English 'if'. [Soon, we'll *prove* the following *equivalence*: $\ulcorner p \to q \urcorner \dashv\vDash \ulcorner {\sim} p \vee q \urcorner$.]

Chapter 3 — Semantics of LSL: Truth Functions VI

p	q	$p \leftrightarrow q$
⊤	⊤	⊤
⊤	⊥	⊥
⊥	⊤	⊥
⊥	⊥	⊤

- Our truth-functional ↔ is also farther from the English 'if and only if'. An LSL biconditional is true iff both sides have the same truth value.
- Consider these two biconditionals. [M = 'the moon's made of green cheese', U = 'there are unicorns', E = 'life exists on Earth', and S = 'the sky is blue']
  - The moon is made of green cheese if and only if there are unicorns.
  - Life exists on earth if and only if the sky is blue.
- The LSL translations of these sentences are true. $M \leftrightarrow U$ is true because M and U are false. $E \leftrightarrow S$ is true because E and S are true. This seems to deviate from the English 'iff'. Soon, we'll *prove* the following:

$$\ulcorner p \leftrightarrow q \urcorner \dashv\vDash \ulcorner (p \,\&\, q) \vee ({\sim} p \,\&\, {\sim} q) \urcorner$$

Chapter 3 — Semantics of LSL: Truth Functions VII

- If our truth-functional semantics for '→' doesn't perfectly capture the English meaning of 'if … then …', then why do we define it this way?
- The answer has two parts. First, our semantics is *truth-functional*. This is an *idealization* — it yields the *simplest* ("Newtonian") semantics.
- And, there are only $2^4 = 16$ possible binary truth-functions. Why?
- So, unless one of the *other* 15 binary truth-functions is *closer* to the English conditional than '→' is, it's *the best we can do, truth-functionally*.
- More importantly, there are certain *logical properties* that the conditional *must* have. It can be shown that our definition of '→' is the *only* binary truth-function which satisfies all three of the following:
  (1) *Modus Ponens* [p and $\ulcorner p \to q \urcorner \therefore q$] is a valid sentential form.
  (2) Affirming the consequent [q and $\ulcorner p \to q \urcorner \therefore p$] is *not* a valid form.
  (3) All sentences of the form $\ulcorner p \to p \urcorner$ are logical truths.

Chapter 3 — Semantics of LSL: Truth Functions VIII

- Here are all of the 16 possible binary truth-functions. I've given them all names or descriptions. [Only a few of these names were made up by me.]

p	q	$\top$	NAND	$\to$	$\sim p$	FI ($\leftarrow$)	$\sim q$	$\leftrightarrow$	NOR	$\vee$	NIFF	q	NFI	p	NIF	&	$\bot$
$\top$	$\top$	$\top$	$\bot$	$\top$	$\bot$	$\top$	$\bot$	$\top$	$\bot$	$\top$	$\bot$	$\top$	$\bot$	$\top$	$\bot$	$\top$	$\bot$
$\top$	$\bot$	$\top$	$\top$	$\bot$	$\bot$	$\top$	$\top$	$\bot$	$\bot$	$\top$	$\top$	$\bot$	$\bot$	$\top$	$\top$	$\bot$	$\bot$
$\bot$	$\top$	$\top$	$\top$	$\top$	$\top$	$\bot$	$\bot$	$\bot$	$\bot$	$\top$	$\top$	$\top$	$\top$	$\bot$	$\bot$	$\bot$	$\bot$
$\bot$	$\bot$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
(1)?				Yes													
(2)?				Yes													
(3)?				Yes													

- Exercise: fill-in the three rows at the bottom (except for →, which I have done for you already) concerning (1), (2), and (3) from the previous slide.
- You should be able to do this pretty soon (within the next week) …

Chapter 3 — Semantics of LSL: Additional Remarks on →

- Above, I explained *why* our conditional → behaves "like a disjunction":
  1. We want a *truth-functional* semantics for →. This is a simplifying *idealization*. Truth-functional semantics are the simplest compositional semantics for sentential logic. [A "Newtonian" semantic model.]
  2. Given (1), the *only* way to define → is *our* way, since it's the *only* binary truth-function that has the following three essential *logical* properties:
     (i) *Modus Ponens* [p and $\ulcorner p \to q \urcorner \therefore q$] is a valid sentential form.
     (ii) Affirming the consequent [q and $\ulcorner p \to q \urcorner \therefore p$] is *not* a valid form.
     (iii) All sentences of the form $\ulcorner p \to p \urcorner$ are logical truths.
- There are *non*-truth-functional semantics for the English conditional.
- These may be "closer" to the English *meaning* of "if". But, they agree with our semantics for →, when it comes to the crucial *logical* properties (i)–(iii). Indeed, our → captures *most* of the (intuitive) *logical* properties of "if".

Interpretations and the Relation of Logical Consequence

- An *interpretation* of an LSL formula p is an assignment of truth-values to all of the sentence letters in p — *i.e.*, a row in p's truth-table.
- A formula p is a *logical consequence* of a set of formulae S [written $S \vDash p$] just in case there is no interpretation (*i.e.*, no row in the joint truth-table of S and p) on which all the members of S are $\top$ but p is $\bot$.
- $S \vDash p$ is another way of saying that the argument from S to p is *valid*.
- Two LSL sentences p and q are said to be *logically equivalent* [written $p \dashv\vDash q$] iff they have the same truth-value on all (joint) interpretations.
- That is, p and q are logically equivalent iff *both* $p \vDash q$ *and* $q \vDash p$.
- I will often express $\ulcorner p \vDash q \urcorner$ by saying that $\ulcorner p$ *entails* $q \urcorner$. This is easier than saying that $\ulcorner q$ is a logical consequence of $p \urcorner$.
- The logical consequence relation $\vDash$ is our central theoretical relation.

Logical Truth, Logical Falsity, and Contingency: Definitions

- A statement is said to be logically true (or tautologous) if it is $\top$ on all interpretations. *E.g.*, any statement of the form $p \leftrightarrow p$ is tautological.

p	p	$\leftrightarrow$	p
$\top$	$\top$	$\top$	$\top$
$\bot$	$\bot$	$\top$	$\bot$

- A statement is logically false (or self-contradictory) if it is $\bot$ on all interpretations. *E.g.*, any statement of the form $p \,\&\, {\sim}p$ is logically false:

p	p	$\&$	$\sim$	p
$\top$	$\top$	$\bot$	$\bot$	$\top$
$\bot$	$\bot$	$\bot$	$\top$	$\bot$

- A statement is contingent if it is *neither* tautological *nor* self-contradictory. Example: 'A' (or *any* basic sentence) is contingent.

A	A
$\top$	$\top$
$\bot$	$\bot$

Logical Truth, Logical Falsity, and Contingency: Problems

- Classify the following statements as logically true (tautologous), logically false (self-contradictory), or contingent:

1. $N \to (N \to N)$
2. $(G \to G) \to G$
3. $(S \to R) \,\&\, (S \,\&\, {\sim}R)$
4. $((E \to F) \to F) \to E$
6. $(M \to P) \vee (P \to M)$
11. $[(Q \to P) \,\&\, ({\sim}Q \to R)] \,\&\, {\sim}(P \vee R)$
12. $[(H \to N) \,\&\, (T \to N)] \to [(H \vee T) \to N]$
15. $[(F \vee E) \,\&\, (G \vee H)] \leftrightarrow [(G \,\&\, E) \vee (F \,\&\, H)]$

Equivalence, Contradictoriness, Consistency, and Inconsistency

- Statements p and q are equivalent [$p \dashv\vDash q$] if they have the same truth-value on all interpretations. For instance, '$A \to B$' and '${\sim}A \vee B$'.

A	B	A	$\to$	B	$\sim$	A	$\vee$	B
$\top$	$\top$	$\top$	$\top$	$\top$	$\bot$	$\top$	$\top$	$\top$
$\top$	$\bot$	$\top$	$\bot$	$\bot$	$\bot$	$\top$	$\bot$	$\bot$
$\bot$	$\top$	$\bot$	$\top$	$\top$	$\top$	$\bot$	$\top$	$\top$
$\bot$	$\bot$	$\bot$	$\top$	$\bot$	$\top$	$\bot$	$\top$	$\bot$

- Statements p and q are contradictory [$p \dashv\vDash {\sim}q$] if they have opposite truth-values on all interpretations. For instance, '$A \to B$' and '$A \,\&\, {\sim}B$'.

A	B	A	$\to$	B	A	$\&$	$\sim$	B
$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\bot$	$\bot$	$\top$
$\top$	$\bot$	$\top$	$\bot$	$\bot$	$\top$	$\top$	$\top$	$\bot$
$\bot$	$\top$	$\bot$	$\top$	$\top$	$\bot$	$\bot$	$\bot$	$\top$
$\bot$	$\bot$	$\bot$	$\top$	$\bot$	$\bot$	$\bot$	$\top$	$\bot$

- Statements p and q are inconsistent [$p \vDash {\sim}q$] if there is no interpretation on which they are both true. For instance, '$A \leftrightarrow B$' and '$A \,\&\, {\sim}B$' are inconsistent [Note: they are *not* contradictory!].

A	B	A	$\leftrightarrow$	B	A	$\&$	$\sim$	B
$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\bot$	$\bot$	$\top$
$\top$	$\bot$	$\top$	$\bot$	$\bot$	$\top$	$\top$	$\top$	$\bot$
$\bot$	$\top$	$\bot$	$\bot$	$\top$	$\bot$	$\bot$	$\bot$	$\top$
$\bot$	$\bot$	$\bot$	$\top$	$\bot$	$\bot$	$\bot$	$\top$	$\bot$

- Statements p and q are consistent [$p \nvDash {\sim}q$] if there's an interpretation on which they are both true. *E.g.*, '$A \,\&\, B$' and '$A \vee B$' are consistent:

A	B	A	$\&$	B	A	$\vee$	B
$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$
$\top$	$\bot$	$\top$	$\bot$	$\bot$	$\top$	$\top$	$\bot$
$\bot$	$\top$	$\bot$	$\bot$	$\top$	$\bot$	$\top$	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$

Semantic Equivalence, Contradictoriness, *etc.*: Relationships

- What are the logical relationships between $\ulcorner p$ and q are equivalent$\urcorner$, $\ulcorner p$ and q are consistent$\urcorner$, $\ulcorner p$ and q are contradictory$\urcorner$, and $\ulcorner p$ and q are inconsistent$\urcorner$? That is, which of these entails which (and which don't)?

Equivalent	Contradictory
$\Downarrow$? $\Uparrow$	$\Downarrow$? $\Uparrow$
Consistent	Inconsistent

- Answers:
  1. Equivalent $\nRightarrow$ Consistent (*example*?)
  2. Consistent $\nRightarrow$ Equivalent (*example*?)
  3. Contradictory $\Rightarrow$ Inconsistent (*why*?)
  4. Inconsistent $\nRightarrow$ Contradictory (*example*?)

Semantic Equivalence: Example #1

- Recall that $\ulcorner p$ unless $q\urcorner$ translates in LSL as $\ulcorner {\sim} q \to p\urcorner$.
- We've said that we can also translate $\ulcorner p$ unless $q\urcorner$ as $\ulcorner p \vee q\urcorner$.
- This is because $\ulcorner {\sim} q \to p\urcorner$ is *semantically equivalent to* $\ulcorner p \vee q\urcorner$. We may demonstrate this, using the following joint truth-table.

p	q	${\sim}q$	$\to$	p	$p \vee q$
$\top$	$\top$	$\bot$	$\top$	$\top$	$\top$
$\top$	$\bot$	$\top$	$\top$	$\top$	$\top$
$\bot$	$\top$	$\bot$	$\top$	$\bot$	$\top$
$\bot$	$\bot$	$\top$	$\bot$	$\bot$	$\bot$

- The truth-tables of $\ulcorner p \vee q\urcorner$ and $\ulcorner {\sim} q \to p\urcorner$ are the same.
- Thus, ${\sim}q \to p \mathrel{=\!\!|\!\vDash} p \vee q$.

Semantic Equivalence: Example #2

- $\ulcorner p \leftrightarrow q\urcorner$ is an *abbreviation* for $\ulcorner (p \to q) \,\&\, (q \to p)\urcorner$.
- The following truth-table shows it is a *legitimate* abbreviation:

p	q	$(p \to q)$	$\&$	$(q \to p)$	$p \leftrightarrow q$
$\top$	$\top$	$\top$	$\top$	$\top$	$\top$
$\top$	$\bot$	$\bot$	$\bot$	$\top$	$\bot$
$\bot$	$\top$	$\top$	$\bot$	$\bot$	$\bot$
$\bot$	$\bot$	$\top$	$\top$	$\top$	$\top$

- $\ulcorner p \leftrightarrow q\urcorner$ and $\ulcorner (p \to q) \,\&\, (q \to p)\urcorner$ have the same truth-table.
- Thus, $p \leftrightarrow q \mathrel{=\!\!|\!\vDash} (p \to q) \,\&\, (q \to p)$.

Semantic Equivalence: Example #3

- Intuitively, the truth-conditions for *exclusive or* ($\oplus$) are such that $\ulcorner p \oplus q\urcorner$ is true if and only if *exactly* one of p or q is true.
- I said that we could say something equivalent to this using our $\vee$, $\&$, and $\sim$. Specifically, I said $p \oplus q \mathrel{=\!\!|\!\vDash} (p \vee q) \,\&\, {\sim}(p \,\&\, q)$.
- The following truth-table shows that this is correct:

p	q	$(p \vee q)$	$\&$	${\sim}(p \,\&\, q)$	$p \oplus q$
$\top$	$\top$	$\top$	$\bot$	$\bot$	$\bot$
$\top$	$\bot$	$\top$	$\top$	$\top$	$\top$
$\bot$	$\top$	$\top$	$\top$	$\top$	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\top$	$\bot$

- $\ulcorner p \oplus q\urcorner$ and $\ulcorner (p \vee q) \,\&\, {\sim}(p \,\&\, q)\urcorner$ have the same truth-table.

Equivalence, Contradictoriness, *etc.*: Some Problems

- Use truth-tables to determine whether the following pairs of statements are semantically equivalent, contradictory, consistent, or inconsistent.
  1. '$F \,\&\, M$' and '$\sim(F \vee M)$'
  2. '$R \vee \sim S$' and '$S \,\&\, \sim R$'
  3. '$H \leftrightarrow \sim G$' and '$(G \,\&\, H) \vee (\sim G \,\&\, \sim H)$'
  4. '$N \,\&\, (A \vee \sim E)$' and '$\sim A \,\&\, (E \vee \sim N)$'
  5. '$W \leftrightarrow (B \,\&\, T)$' and '$W \,\&\, (T \to \sim B)$'
  6. '$R \,\&\, (Q \vee S)$' and '$(S \vee R) \,\&\, (Q \vee R)$'
  7. '$Z \,\&\, (C \leftrightarrow P)$' and '$C \leftrightarrow (Z \,\&\, \sim P)$'
  8. '$Q \to \sim(K \vee F)$' and '$(K \,\&\, Q) \vee (F \,\&\, Q)$'

Some More Semantic Equivalences

- Here is a simultaneous truth-table which establishes that

$$A \leftrightarrow B \dashv\vDash (A \,\&\, B) \vee (\sim A \,\&\, \sim B)$$

A	B	A	$\leftrightarrow$	B	$(A$	$\&$	$B)$	$\vee$	$(\sim$	A	$\&$	$\sim$	$B)$
⊤	⊤	⊤	⊤	⊤	⊤	⊤	⊤	⊤	⊥	⊤	⊥	⊥	⊤
⊤	⊥	⊤	⊥	⊥	⊤	⊥	⊥	⊥	⊥	⊤	⊥	⊤	⊥
⊥	⊤	⊥	⊥	⊤	⊥	⊥	⊤	⊥	⊤	⊥	⊥	⊥	⊤
⊥	⊥	⊥	⊤	⊥	⊥	⊥	⊥	⊤	⊤	⊥	⊤	⊤	⊥

- Can you prove the following equivalences with truth-tables?
  - $\sim(A \,\&\, B) \dashv\vDash \sim A \vee \sim B$
  - $\sim(A \vee B) \dashv\vDash \sim A \,\&\, \sim B$
  - $A \dashv\vDash (A \,\&\, B) \vee (A \,\&\, \sim B)$
  - $A \dashv\vDash A \,\&\, (B \to B)$
  - $A \dashv\vDash A \vee (B \,\&\, \sim B)$

A More Complicated Equivalence (Distributivity)

- The following simultaneous truth-table establishes that

$$p \,\&\, (q \vee r) \dashv\vDash (p \,\&\, q) \vee (p \,\&\, r)$$

p	q	r	p	$\&$	$(q \vee r)$	$(p \,\&\, q)$	$\vee$	$(p \,\&\, r)$
⊤	⊤	⊤	⊤	⊤	⊤	⊤	⊤	⊤
⊤	⊤	⊥	⊤	⊤	⊤	⊤	⊤	⊥
⊤	⊥	⊤	⊤	⊤	⊤	⊥	⊤	⊤
⊤	⊥	⊥	⊤	⊥	⊥	⊥	⊥	⊥
⊥	⊤	⊤	⊥	⊥	⊤	⊥	⊥	⊥
⊥	⊤	⊥	⊥	⊥	⊤	⊥	⊥	⊥
⊥	⊥	⊤	⊥	⊥	⊤	⊥	⊥	⊥
⊥	⊥	⊥	⊥	⊥	⊥	⊥	⊥	⊥

- This is *distributivity* of & over $\vee$. It also works for $\vee$ over &.

The Exhaustive Truth-Table Method for Testing Validity

- Remember, an argument is valid if it is *impossible* for its premises to be true while its conclusion is false. Let $p_1, \ldots, p_n$ be the premises of a LSL argument, and let q be the conclusion of the argument. Then, we have:

$$\begin{array}{l} p_1 \\ \vdots \\ p_n \\ \hline \therefore q \end{array}$$

is valid if and only if there is no row in the simultaneous truth-table of $p_1, \ldots, p_n$, and q which looks like the following:

atoms premises conclusion

$\cdots$	p_1	$\cdots$	p_n	q
$\cdots$	⊤	⊤	⊤	⊥

- We will use simultaneous truth-tables to prove validities and invalidities. For example, consider the following valid argument:

$$\begin{array}{l} A \\ \underline{A \to B} \\ \therefore B \end{array}$$

atoms		premises				conclusion
A	B	A	A	$\to$	B	B
⊤	⊤	⊤	⊤	⊤	⊤	⊤
⊤	⊥	⊤	⊤	⊥	⊥	⊥
⊥	⊤	⊥	⊥	⊤	⊤	⊤
⊥	⊥	⊥	⊥	⊤	⊥	⊥

☞ VALID — there is no row in which A and $A \to B$ are both ⊤, but B is ⊥.

- In general, we'll use the following procedure for evaluating arguments:
  1. Translate and symbolize the the argument (if given in English).
  2. Write out the symbolized argument (as above).
  3. Draw a simultaneous truth-table for the symbolized argument, outlining the columns representing the premises and conclusion.
  4. Is there a row of the table in which all premises are ⊤ but the conclusion is ⊥? If so, the argument is invalid; if not, it's valid.
- We will practice this on examples. But, first, a "short-cut" method.