Announcements & Such

- *Black Roots.*
- Administrative Stuff
  - HW #4 resubs are still being graded. Stay tuned…
  - **HW #6 is due Thurs. Final HW assignment! *LMPL Proofs.***
  - **Next week, I *will* be holding lectures. I will use them for both review, and for some interesting "logic beyond 12A" topics.**
- Today: Chapter 6 — Natural Deductions in LMPL
  - Introduction and Elimination rules for the quantifiers.
  - Sequents and Theorems (SI/TI) for the quantifiers.
  - **Lots of proofs in LMPL**!
- **Next**: Two-Place predicates (*i.e.*, *binary relations*) — "L2PL".

The Rule of ∃-Elimination: Some Background

- It is useful to think of an existential claim $\ulcorner(\exists v)\phi v\urcorner$ as a *disjunction* which asserts that the predicate expression ϕ is satisfied by *at least one* object in the domain (*i.e.*, that the disjunction $\ulcorner\phi a \vee (\phi b \vee (\phi c \vee \ldots))\urcorner$ is true).
- In this way, we would expect the elimination rule for ∃ to be similar to the elimination rule for ∨. That is, we'd expect the ∃E rule to be similar to the ∨E rule. Indeed, this is the case. It's best to start with a simple example.
- Consider the following *legitimate* elimination of an existential claim:

Problem is: (∃x)(Fx&Gx) ⊢ (∃x)Fx

1	(1)	(∃x)(Fx&Gx)	Premise
2	(2)	Fa&Ga	Assumption
2	(3)	Fa	2 &E
2	(4)	(∃x)Fx	3 ∃I
1	(5)	(∃x)Fx	1,2,4 ∃E

The Rule of ∃-Elimination: II

- To derive a sentence $\mathscr{P}$ using the ∃E rule (with some existential sentence $\ulcorner(\exists v)\phi v\urcorner$), we must first *assume* an *instance* $\phi\tau$ of $\ulcorner(\exists v)\phi v\urcorner$.
- If we can deduce $\mathscr{P}$ from this assumed instance $\phi\tau$ — *using **generalizable** reasoning* — then we may infer $\mathscr{P}$ *outright.*
- It is because our reasoning from the *instance* $\phi\tau$ of $\ulcorner(\exists v)\phi v\urcorner$ to $\mathscr{P}$ *does not depend on our choice of constant* τ (*i.e.*, that our reasoning from $\phi\tau$ to $\mathscr{P}$ is *generalizable*) that makes this inference valid.
- When our reasoning is generalizable in this sense, it's as if we are showing that $\mathscr{P}$ can be deduced from *any* instance $\phi\tau$ of $\ulcorner(\exists v)\phi v\urcorner$.
- As such, this is just like showing that $\mathscr{P}$ can be deduced from *any disjunct* of the disjunction $\ulcorner\phi a \vee (\phi b \vee (\phi c \vee \ldots))\urcorner$. And, this is just like ∨E reasoning (except that ∃E only requires *one* assumption).

The Rule of ∃-Elimination: III

- Here's an *il*legitimate "∃-Elimination" step:

1	(1)	(∃x)Fx	Premise	
2	(2)	Ga	Premise	
3	(3)	Fa	Assumption	
2,3	(4)	Fa&Ga	2,3 &I	
2,3	(5)	(∃x)(Fx&Gx)	4 ∃I	
1,2	(6)	(∃x)(Fx&Gx)	1,3,5 ∃E	NO!!

- This is *not* a valid inference: $(\exists x)Fx, Ga \nvDash (\exists x)(Fx \,\&\, Gx)$!
- So, what went wrong here? The problem is that the inference to '$(\exists x)(Fx \,\&\, Gx)$' at line (5) does *not* use *generalizable* reasoning.
- We can *not* legitimately infer '$(\exists x)(Fx \,\&\, Gx)$' at line (5) from an *arbitrary instance* $\ulcorner F\tau\urcorner$ of '$(\exists x)Fx$'. We *must* assume '*F**a***' *in particular* at line (3) in order to deduce '$(\exists x)(Fx \,\&\, Gx)$' at line (5).

The Rule of ∃-Elimination: Official Definition

∃-Elimination: If ⌜$(\exists v)\phi v$⌝ occurs at i depending on $a_1, \ldots, a_n$, an instance $\phi\tau$ of ⌜$(\exists v)\phi v$⌝ is *assumed* at j, and $\mathscr{P}$ is inferred at k depending on $b_1, \ldots, b_u$, then at line m we may infer $\mathscr{P}$, with label 'i, j, k ∃E' and dependencies $\{a_1, \ldots, a_n\} \cup \{b_1, \ldots, b_u\}/j$:

$a_1, \ldots, a_n$	(i)	$(\exists v)\phi v$	
	⋮		
j	(j)	$\phi\tau$	Assumption
	⋮		
$b_1, \ldots, b_u$	(k)	$\mathscr{P}$	
	⋮		
$\{a_1, \ldots, a_n\} \cup \{b_1, \ldots, b_u\}/j$	(m)	$\mathscr{P}$	i, j, k ∃E

Provided that ***all four*** of the following conditions are met:

- τ (in $\phi\tau$) replaces ***every*** occurrence of v in ϕv. [avoids fallacies]
- τ ***does not occur in*** ⌜$(\exists v)\phi v$⌝. [generalizability]
- τ ***does not occur in*** $\mathscr{P}$. [generalizability]
- τ ***does not occur in*** *any* of $b_1, \ldots, b_u$, except (possibly) $\phi\tau$ itself. [generalizability]

The Rule of ∃-Elimination: Nine Examples

- Here are 9 examples of proofs involving all four quantifier rules.

1. $(\exists x){\sim}Fx \vdash {\sim}(\forall x)Fx$ [p. 200, example 5]
2. $(\exists x)(Fx \to A) \vdash (\forall x)Fx \to A$ [p. 201, example 6]
3. $(\forall x)(\forall y)(Gy \to Fx) \vdash (\forall x)[(\exists y)Gy \to Fx]$ [p. 203, I. # 19 ⇒]
4. $(\exists x)[Fx \to (\forall y)Gy] \vdash (\exists x)(\forall y)(Fx \to Gy)$ [p. 203, I. # 20 ⇐]
5. $A \vee (\exists x)Fx \vdash (\exists x)(A \vee Fx)$ [p. 203, II. # 2 ⇐]
6. $(\exists x)(Fx \,\&\, {\sim}Fx) \vdash (\forall x)(Gx \,\&\, {\sim}Gx)$ [p. 203, I. # 12 ⇒]
7. $(\forall x)[Fx \to (\forall y){\sim}Fy] \vdash {\sim}(\exists x)Fx$ [p. 203, I. # 5]
8. $(\forall x)(\exists y)(Fx \,\&\, Gy) \vdash (\exists y)(\forall x)(Fx \,\&\, Gy)$ [p. 201, example 7]
9. $(\exists y)(\forall x)(Fx \,\&\, Gy) \vdash (\forall x)(\exists y)(Fx \,\&\, Gy)$ [other direction]

Proof of (1)

Problem is: (∃x)~Fx ⊢ ~(∀x)Fx

1	(1)	(∃x)~Fx	Premise
2	(2)	(∀x)Fx	Assumption
3	(3)	~Fa	Assumption
2	(4)	Fa	2 ∀E
2,3	(5)	Λ	3,4 ~E
1,2	(6)	Λ	1,3,5 ∃E
1	(7)	~(∀x)Fx	2,6 ~I

Proof of (2)

Problem is: (∃x)(Fx→A) ⊢ (∀x)Fx→A

1	(1)	(∃x)(Fx→A)	Premise
2	(2)	(∀x)Fx	Assumption
3	(3)	Fa→A	Assumption
2	(4)	Fa	2 ∀E
2,3	(5)	A	3,4 →E
1,2	(6)	A	1,3,5 ∃E
1	(7)	(∀x)Fx→A	2,6 →I

Proof of (3)

Problem is: (∀x)(∀y)(Gy→Fx) ⊦ (∀x)((∃y)Gy→Fx)

1	(1)	(∀x)(∀y)(Gy→Fx)	Premise
2	(2)	(∃y)Gy	Assumption
3	(3)	Gb	Assumption
1	(4)	(∀y)(Gy→Fa)	1 ∀E
1	(5)	Gb→Fa	4 ∀E
1,3	(6)	Fa	5,3 →E
1,2	(7)	Fa	2,3,6 ∃E
1	(8)	(∃y)Gy→Fa	2,7 →I
1	(9)	(∀x)((∃y)Gy→Fx)	8 ∀I

Proof of (4)

Problem is: (∃x)(Fx→(∀y)Gy) ⊦ (∃x)(∀y)(Fx→Gy)

1	(1)	(∃x)(Fx→(∀y)Gy)	Premise
2	(2)	Fa→(∀y)Gy	Assumption
3	(3)	Fa	Assumption
2,3	(4)	(∀y)Gy	2,3 →E
2,3	(5)	Gb	4 ∀E
2	(6)	Fa→Gb	3,5 →I
2	(7)	(∀y)(Fa→Gy)	6 ∀I
2	(8)	(∃x)(∀y)(Fx→Gy)	7 ∃I
1	(9)	(∃x)(∀y)(Fx→Gy)	1,2,8 ∃E

Proof of (5)

Problem is: A∨(∃x)Fx ⊦ (∃x)(A∨Fx)

1	(1)	A∨(∃x)Fx	Premise
2	(2)	A	Assumption
2	(3)	A∨Fa	2 ∨I
2	(4)	(∃x)(A∨Fx)	3 ∃I
5	(5)	(∃x)Fx	Assumption
6	(6)	Fa	Assumption
6	(7)	A∨Fa	6 ∨I
6	(8)	(∃x)(A∨Fx)	7 ∃I
5	(9)	(∃x)(A∨Fx)	5,6,8 ∃E
1	(10)	(∃x)(A∨Fx)	1,2,4,5,9 ∨E

Proof of (6)

Problem is: (∃x)(Fx&~Fx) ⊦ (∀x)(Gx&~Gx)

1	(1)	(∃x)(Fx&~Fx)	Premise
2	(2)	Fa&~Fa	Assumption
3	(3)	~Gb	Assumption
2	(4)	~Fa	2 &E
2	(5)	Fa	2 &E
2	(6)	Λ	4,5 ~E
2	(7)	~~Gb	3,6 ~I
2	(8)	Gb	7 DN
9	(9)	Gb	Assumption
2	(10)	~Gb	9,6 ~I
2	(11)	Gb&~Gb	8,10 &I
2	(12)	(∀x)(Gx&~Gx)	11 ∀I
1	(13)	(∀x)(Gx&~Gx)	1,2,12 ∃E

Proof of (7)

Problem is: (∀x)(Fx→(∀y)~Fy) ⊢ ~(∃x)Fx

1	(1)	(∀x)(Fx→(∀y)~Fy)	Premise
2	(2)	(∃x)Fx	Assumption
3	(3)	Fa	Assumption
1	(4)	Fa→(∀y)~Fy	1 ∀E
1,3	(5)	(∀y)~Fy	4,3 →E
1,3	(6)	~Fa	5 ∀E
1,3	(7)	⋏	6,3 ~E
1,2	(8)	⋏	2,3,7 ∃E
1	(9)	~(∃x)Fx	2,8 ~I

Proof of (8)

Problem is: (∀x)(∃y)(Fx&Gy) ⊢ (∃y)(∀x)(Fx&Gy)

1	(1)	(∀x)(∃y)(Fx&Gy)	Premise
1	(2)	(∃y)(Fa&Gy)	1 ∀E
3	(3)	Fa&Gb	Assumption
1	(4)	(∃y)(Fc&Gy)	1 ∀E
5	(5)	Fc&Gd	Assumption
5	(6)	Fc	5 &E
1	(7)	Fc	4,5,6 ∃E
3	(8)	Gb	3 &E
1,3	(9)	Fc&Gb	7,8 &I
1,3	(10)	(∀x)(Fx&Gb)	9 ∀I
1,3	(11)	(∃y)(∀x)(Fx&Gy)	10 ∃I
1	(12)	(∃y)(∀x)(Fx&Gy)	2,3,11 ∃E

Proof of (9)

Problem is: (∃y)(∀x)(Fx&Gy) ⊢ (∀x)(∃y)(Fx&Gy)

1	(1)	(∃y)(∀x)(Fx&Gy)	Premise
2	(2)	(∀x)(Fx&Gb)	Assumption
2	(3)	Fa&Gb	2 ∀E
2	(4)	(∃y)(Fa&Gy)	3 ∃I
1	(5)	(∃y)(Fa&Gy)	1,2,4 ∃E
1	(6)	(∀x)(∃y)(Fx&Gy)	5 ∀I

Two LMPL Extensions of Sequent Introduction

- Here are two additions to our list of SI sequents:

(QS) One can infer ⌜$(\forall x){\sim}\phi x$⌝ from (the *logically equivalent* sentence) ⌜${\sim}(\exists x)\phi x$⌝, and *vice versa*; and, that one can infer ⌜$(\exists x){\sim}\phi x$⌝ from (the *logically equivalent*) ⌜${\sim}(\forall x)\phi x$⌝, and *vice versa*.

$$(\forall x){\sim}\phi x \dashv\vdash {\sim}(\exists x)\phi x\text{; and, } (\exists x){\sim}\phi x \dashv\vdash {\sim}(\forall x)\phi x \qquad \text{(QS)}$$

(AV) One can infer a *closed* LMPL sentence ψ from (the *logically equivalent* sentence) ψ', and *vice versa*, where ψ and ψ' are *alphabetic variants*. Two formulas are *alphabetic variants* if and only if they differ *only* in a (conventional) choice of individual *variable* letters (*not* kosher for constants!). *E.g.*, '$(\forall x)Fx$' and '$(\forall y)Fy$' are (closed) *alphabetic variants*, because they differ *only* in which individual variable ('x' or 'y') is used, but they have the same *logical* (*i.e.*, *syntactical*) *structure*.

$$\psi \dashv\vdash \psi' \qquad \text{(AV)}$$

Our (New) Official List of Sequents and Theorems (see pp. 123, 204, and 206)

$A \vee B, \sim A \vdash B$; or; $A \vee B, \sim B \vdash A$	(DS)	$A \rightarrow B \dashv\vdash \sim A \vee B$	(Imp)
$A \rightarrow B, \sim B \vdash \sim A$	(MT)	$\sim(A \rightarrow B) \dashv\vdash A \,\&\, \sim B$	(Neg-Imp)
$A \vdash B \rightarrow A$	(PMI)	$A \,\&\, (B \vee C) \dashv\vdash (A \,\&\, B) \vee (A \,\&\, C)$	(Dist)
$\sim A \vdash A \rightarrow B$	(PMI)	$A \vee (B \,\&\, C) \dashv\vdash (A \vee B) \,\&\, (A \vee C)$	(Dist)
$A \vdash \sim\sim A$	(DN$^+$)	$\curlywedge \vdash A$	(EFQ, or $\curlywedge$E)
$\sim(A \,\&\, B) \dashv\vdash \sim A \vee \sim B$	(DEM)	$A * B \vdash B * A$	(Com)
$\sim(A \vee B) \dashv\vdash \sim A \,\&\, \sim B$	(DEM)	$\sim\sim A * \sim\sim B \dashv\vdash A * B$	(SDN)
$\sim(\sim A \vee \sim B) \dashv\vdash A \,\&\, B$	(DEM)	$A * B \dashv\vdash \sim\sim A * B \dashv\vdash A * \sim\sim B$	(SDN)
$\sim(\sim A \,\&\, \sim B) \dashv\vdash A \vee B$	(DEM)	$\vdash A \vee \sim A$	(LEM)
$(\forall x)\sim\phi x \dashv\vdash \sim(\exists x)\phi x$	(QS)	$(\exists x)\sim\phi x \dashv\vdash \sim(\forall x)\phi x$	(QS)
		$\psi \dashv\vdash \psi'$	(AV)

In (Com), '$*$' can be any binary connective *except* '$\rightarrow$'. In (SDN), '$*$' can be *any* binary connective. In (AV), ψ must be *closed*, and ψ' must be an *alphabetic variant* of ψ.

The Value of (QS) — Its Four Simplest Instances

$(\forall x)\sim Fx \vdash \sim(\exists x)Fx$			
1	(1)	$(\forall x)\sim Fx$	Premise
2	(2)	$(\exists x)Fx$	Ass
3	(3)	Fa	Ass
1	(4)	$\sim Fa$	1 $\forall$E
1,3	(5)	$\curlywedge$	4,3 ~E
1,2	(6)	$\curlywedge$	2,3,5 $\exists$E
1	(7)	$\sim(\exists x)Fx$	2,6 ~I

$\sim(\exists x)Fx \vdash (\forall x)\sim Fx$			
1	(1)	$\sim(\exists x)Fx$	Premise
2	(2)	Fa	Ass
2	(3)	$(\exists x)Fx$	2 $\exists$I
1,2	(4)	$\curlywedge$	1,3 ~E
1	(5)	$\sim Fa$	2,4 ~I
1	(6)	$(\forall x)\sim Fx$	5 $\forall$I

$(\exists x)\sim Fx \vdash \sim(\forall x)Fx$			
1	(1)	$(\exists x)\sim Fx$	Premise
2	(2)	$(\forall x)Fx$	Ass
3	(3)	$\sim Fa$	Ass
2	(4)	Fa	2 $\forall$E
2,3	(5)	$\curlywedge$	3,4 ~E
1,2	(6)	$\curlywedge$	1,3,5 $\exists$E
1	(7)	$\sim(\forall x)Fx$	2,6 ~I

$\sim(\forall x)Fx \vdash (\exists x)\sim Fx$			
1	(1)	$\sim(\forall x)Fx$	Premise
2	(2)	$\sim(\exists x)\sim Fx$	Ass
3	(3)	$\sim Fa$	Ass
3	(4)	$(\exists x)\sim Fx$	3 $\exists$I
2,3	(5)	$\curlywedge$	2,4 ~E
2	(6)	$\sim\sim Fa$	3,5 ~I
2	(7)	Fa	6 DN
2	(8)	$(\forall x)Fx$	7 $\forall$I
1,2	(9)	$\curlywedge$	1,8 ~E
1	(10)	$\sim\sim(\exists x)\sim Fx$	2,9 ~I
1	(11)	$(\exists x)\sim Fx$	10 DN

Three Examples Involving the LMPL SI Extension (QS)

- Here are three examples of proofs involving SI (QS):
  1. $\sim(\forall x)\sim Fx \vdash (\exists x)Fx$ [p. 207, #7 $\Leftarrow$]
  2. $\sim(\exists x)(Fx \,\&\, Gx) \vee (\exists x)\sim Gx, (\forall y)Gy \vdash (\forall z)(Fz \rightarrow \sim Gz)$ [p. 205, ex. 1]
  3. $(\forall x)Fx \rightarrow A \vdash (\exists x)(Fx \rightarrow A)$ [p. 205, ex. 2]

Proof of (1)

1	(1)	$\sim(\forall x)\sim Fx$	Premise
2	(2)	$\sim(\exists x)Fx$	Assumption
2	(3)	$(\forall x)\sim Fx$	2 SI (QS)
1,2	(4)	$\curlywedge$	1, 3 ~E
1	(5)	$\sim\sim(\exists x)Fx$	2, 4 ~I
1	(6)	$(\exists x)Fx$	5 DN

Proof of (2)

1	(1)	$\sim(\exists x)(Fx \& Gx) \vee (\exists x)\sim Gx$	Premise
2	(2)	$(\forall y)Gy$	Premise
3	(3)	$\sim(\exists x)(Fx \& Gx)$	Assumption
3	(4)	$(\forall x)\sim(Fx \& Gx)$	3 SI (QS)
3	(5)	$\sim(Fa \& Ga)$	4 ∀E
3	(6)	$\sim Fa \vee \sim Ga$	5 SI (DeM)
3	(7)	$Fa \to \sim Ga$	6 SI (Imp)
3	(8)	$(\forall z)(Fz \to \sim Gz)$	7 ∀I
9	(9)	$(\exists x)\sim Gx$	Assumption
10	(10)	$\sim Ga$	Assumption
2	(11)	Ga	2 ∀E
2,10	(12)	$\curlywedge$	10, 11 ~E
2,10	(13)	$(\forall z)(Fz \to \sim Gz)$	12 SI (EFQ)
2,9	(14)	$(\forall z)(Fz \to \sim Gz)$	9, 10, 13 ∃E
1,2	(15)	$(\forall z)(Fz \to \sim Gz)$	1, 3, 8, 9, 14 ∨E

Proof of (3)

Problem is: $(\forall x)Fx \to A \vdash (\exists x)(Fx \to A)$

1	(1)	$(\forall x)Fx \to A$	Premise
1	(2)	$\sim(\forall x)Fx \vee A$	1 SI (Imp)
3	(3)	$\sim(\forall x)Fx$	Assumption
3	(4)	$(\exists x)\sim Fx$	3 SI (QS)
5	(5)	$\sim Fa$	Assumption
5	(6)	$Fa \to A$	5 SI (PMI)
5	(7)	$(\exists x)(Fx \to A)$	6 ∃I
3	(8)	$(\exists x)(Fx \to A)$	4,5,7 ∃E
9	(9)	A	Assumption
9	(10)	$Fa \to A$	9 SI (PMI)
9	(11)	$(\exists x)(Fx \to A)$	10 ∃I
1	(12)	$(\exists x)(Fx \to A)$	2,3,8,9,11 ∨E

The Value of (AV)

- Here are the two simplest instances of (AV):

$(\forall x)Fx \vdash (\forall y)Fy$				$(\exists x)Fx \vdash (\exists y)Fy$			
1	(1)	$(\forall x)Fx$	Premise	1	(1)	$(\exists x)Fx$	Premise
1	(2)	Fa	1 ∀E	2	(2)	Fa	Ass
1	(3)	$(\forall y)Fy$	2 ∀I	2	(3)	$(\exists y)Fy$	2 ∃I
				1	(4)	$(\exists y)Fy$	1,2,3 ∃E

- Here's an (AV)-aided proof of the following sequent

$$(\forall x)Fx, (\forall y)Fy \to (\forall y)Gy \vdash (\forall z)Gz$$

1	(1)	$(\forall x)Fx$	Premise
2	(2)	$(\forall y)Fy \to (\forall y)Gy$	Premise
1	(3)	$(\forall y)Fy$	1 SI (AV)
1,2	(4)	$(\forall y)Gy$	2,3 →E
1,2	(5)	$(\forall z)Gz$	4 SI (AV)