Announcements & Such

- *Steel Pulse.*
- Administrative Stuff
  - HW #4 resubs are still being graded. Stay tuned...
  - **HW #5 resubmission is due today** (follow models on handout).
  - **HW #6 is posted. Final HW assignment! *LMPL Proofs.***
  - **From now on, my office hours are: 4–6pm Tuesdays.**
- Today: Chapter 6 — Natural Deductions in LMPL
  - Introduction and Elimination rules for the quantifiers.
  - Sequents and Theorems (SI/TI) for the quantifiers.
  - **Lots of proofs in LMPL**!
- **Next**: Two-Place predicates (*i.e.*, *binary relations*) — "L2PL".

Natural Deduction Proofs in LMPL

- The natural deduction rules for LMPL will *include* the rules for LSL that we already know (*viz.*, Ass., &E, &I, $\rightarrow$E, $\rightarrow$I, $\sim$E, $\sim$I, DN, $\vee$E, $\vee$I, Df.).
- Plus, we will be *adding* 4 new rules. We will need both introduction and elimination rules for each of the two quantifiers ($\exists$I, $\exists$E, $\forall$I, $\forall$E).
- As in LSL, the system will be *sound and complete* (140A!). That is, $\vdash$ will apply to the same sequents that $\vDash$ does in our semantics for LMPL.
- We begin with the simplest: the introduction rule for $\exists$ ($\exists$I). Intuitively, if we have proved $\phi\tau$ for some individual constant τ, then we may infer that ϕ is true of *something* (*e.g.*, that $(\exists x)\phi x$).
- *E.g.*, if we've proved '$Pa \,\&\, Qa$', we may validly infer '$(\exists x)(Px \,\&\, Qx)$'.
- We may also infer '$(\exists x)(Pa \,\&\, Qx)$' and '$(\exists x)(Px \,\&\, Qa)$' from '$Pa \,\&\, Qa$'.
- These (and similar) considerations lead us to the $\exists$I rule ...

The Rule of ∃-Introduction

Rule of $\exists$-Introduction: For any sentence $\phi\tau$, if $\phi\tau$ has been inferred at line j in a proof, then at line k we may infer $\ulcorner(\exists\nu)\phi\nu\urcorner$, labeling the line 'j $\exists$I' and writing on its left the numbers that occur on the left of j.

$a_1,\ldots,a_n$	(j)	$\phi\tau$	
	$\vdots$		
$a_1,\ldots,a_n$	(k)	$(\exists\nu)\phi\nu$	j $\exists$I

Where $\ulcorner(\exists\nu)\phi\nu\urcorner$ is obtained syntactically from $\phi\tau$ by:

- Replacing ***one or more occurrences*** of τ in $\phi\tau$ by a *single* variable ν.
- Note: the variable ν ***must not already occur in*** the expression $\phi\tau$. [This prevents *double-binding*, *e.g.*, '$(\exists x)(\exists x)(Fx \,\&\, Gx)$'.]
- And, finally, prefixing the quantifier $\ulcorner(\exists\nu)\urcorner$ in front of the resulting expression (which may now have both $\ulcorner\nu\urcorner$s and $\ulcorner\tau\urcorner$s occurring in it).

The Rule of ∀-Elimination

Rule of $\forall$-Elimination: For any sentence $\ulcorner(\forall\nu)\phi\nu\urcorner$ and constant τ, if $\ulcorner(\forall\nu)\phi\nu\urcorner$ has been inferred at a line j, then at line k we may infer $\phi\tau$, labeling the line 'j $\forall$E' and writing on its left the numbers that appear on the left of j.

$a_1,\ldots,a_n$	(j)	$(\forall\nu)\phi\nu$	
	$\vdots$		
$a_1,\ldots,a_n$	(k)	$\phi\tau$	j $\forall$E

Where $\phi\tau$ is obtained syntactically from $\ulcorner(\forall\nu)\phi\nu\urcorner$ by:

- Deleting the quantifier prefix $\ulcorner(\forall\nu)\urcorner$.
- Replacing ***every occurrence*** of ν in the open sentence $\phi\nu$ by ***one and the same*** constant τ. [This prevents *fallacies*, *e.g.*, $(\forall x)(Fx \,\&\, Gx) \nvDash Fa \,\&\, Gb$.]
- Note: since '$\forall$' means *everything*, there are *no* restrictions on *which* individual constant may be used in an application of $\forall$E.

An Example Proof Involving Both ∃I and ∀E

Let's prove that $(\forall x)(Fx \to Gx), Fa \vdash (\exists x)(\sim Gx \to Hx)$.

1	(1)	(∀x)(Fx→Gx)	Premise
2	(2)	Fa	Premise
3	(3)	~Ga	Assumption
4	(4)	~Ha	Assumption
1	(5)	Fa→Ga	1 ∀E
1,2	(6)	Ga	5,2 →E
1,2,3	(7)	⋀	3,6 ~E
1,2,3	(8)	~~Ha	4,7 ~I
1,2,3	(9)	Ha	8 DN
1,2	(10)	~Ga→Ha	3,9 →I
1,2	(11)	(∃x)(~Gx→Hx)	10 ∃I

- This example illustrates a typical pattern in quantificational proofs: quantifiers are removed from the premises using elimination rules, sentential (*viz.*, LSL) rules are applied, and then quantifiers are reintroduced using introduction rules to obtain the conclusion.

The Rule of ∀-Introduction: Some Background

- It is useful to think of a universal claim $\ulcorner(\forall\nu)\phi\nu\urcorner$ as a *conjunction* which asserts that the predicate expression ϕ is satisfied by *all objects* in the domain of discourse (*i.e.*, the conjunction $\ulcorner\phi a \,\&\, (\phi b \,\&\, (\phi c \,\&\, \ldots))\urcorner$ is true).
- So, in order to be able to *introduce* the universal quantifier (*i.e.*, to *legitimately infer* $\ulcorner(\forall\nu)\phi\nu\urcorner$ in a proof), we must be in a position to prove $\phi\tau$, for *any* individual constant τ. This is called *generalizable reasoning*.
- Consider the following *legitimate* introduction of a universal claim:

Problem is: (∀x)(Fx→Gx), (∀x)Fx ⊢ (∀x)Gx

1	(1)	(∀x)(Fx→Gx)	Premise
2	(2)	(∀x)Fx	Premise
1	(3)	Fa→Ga	1 ∀E
2	(4)	Fa	2 ∀E
1,2	(5)	Ga	3,4 →E
1,2	(6)	(∀x)Gx	5 ∀I

The Rule of ∀-Introduction: II

- We can legitimately infer '$(\forall x)Gx$' at line 6 of this proof, because our inference to 'Gb' is *generalizable* — *i.e.*, we could have deduced $\ulcorner G\tau\urcorner$, for *any* individual constant τ — using *exactly parallel* reasoning.
- However, consider the following *il*legitimate "∀-Introduction" step:

1	(1)	(∀x)(Fx→Gx)	Premise	
2	(2)	Fb	Premise	
1	(3)	Fb→Gb	1 ∀E	
1,2	(4)	Gb	2,3 →E	
1,2	(5)	(∀x)Gx	4 ∀I	NO!!

- This is *not* a valid inference, since $(\forall x)(Fx \to Gx), Fb \nvDash (\forall x)Gx$!
- So, what went wrong? The problem is that the inference to 'Gb' at (4) is *not* generalizable. We can *not* deduce $\ulcorner G\tau\urcorner$ — for *any* τ — from the premises '$(\forall x)(Fx \to Gx)$' and 'Fb'. We can *only* infer '$G\boldsymbol{b}$'.

The Rule of ∀-Introduction: III

Rule of ∀-Introduction: For any sentence $\phi\tau$, if $\phi\tau$ has been inferred at a line j, then *provided that* τ *does not occur in any premise or assumption whose line number is on the left at line* j, we may infer $\ulcorner(\forall\nu)\phi\nu\urcorner$ at line k, labeling the line 'j ∀I' and writing on its left the same numbers as occur on the left at line j.

$$\begin{array}{llll} a_1,\ldots,a_n & (j) & \phi\tau & \\ & \vdots & & \\ a_1,\ldots,a_n & (k) & (\forall\nu)\phi\nu & j\ \forall I \end{array}$$

Where $\ulcorner(\forall\nu)\phi\nu\urcorner$ is obtained by:

- Replacing ***every*** occurrence of τ in $\phi\tau$ with ν and prefixing $\ulcorner(\forall\nu)\urcorner$. [Again, 'every' prevents *fallacies*, *e.g.*, $(\forall x)(Fx \to Gx) \nvdash (\forall x)(\forall y)(Fx \to Gy)$.]
- τ ***does not occur in*** any of the formulae $a_1,\ldots,a_n$. [ensures *generalizability*]
- ν ***does not occur in*** $\phi\tau$. [prevents *double-binding*]

The Rule of ∀-Introduction: Four Examples

- Here are four examples of LMPL sequents involving the three quantifier rules we've learned so far (∃I, ∀E, and ∀I).

(1) $(\forall x)(Fx \to Gx) \vdash (\forall x)Fx \to (\forall x)Gx$

(2) $\sim(\exists x)(Fx \,\&\, Gx) \vdash (\forall x)(Fx \to \sim Gx)$

(3) $\sim(\forall x)Fx \vdash (\exists x)\sim Fx$

(4) $(\forall x)[Fx \to (\forall y)Gy] \vdash (\forall x)(\forall y)(Fx \to Gy)$

Proof of (1)

Problem is: (∀x)(Fx→Gx) ⊢ (∀x)Fx→(∀x)Gx

1	(1)	(∀x)(Fx→Gx)	Premise
2	(2)	(∀x)Fx	Assumption
1	(3)	Fa→Ga	1 ∀E
2	(4)	Fa	2 ∀E
1,2	(5)	Ga	3,4 →E
1,2	(6)	(∀x)Gx	5 ∀I
1	(7)	(∀x)Fx→(∀x)Gx	2,6 →I

Proof of (2)

Problem is: ~(∃x)(Fx&Gx) ⊢ (∀x)(Fx→~Gx)

1	(1)	~(∃x)(Fx&Gx)	Premise
2	(2)	Fa	Assumption
3	(3)	Ga	Assumption
2,3	(4)	Fa&Ga	2,3 &I
2,3	(5)	(∃x)(Fx&Gx)	4 ∃I
1,2,3	(6)	Λ	1,5 ~E
1,2	(7)	~Ga	3,6 ~I
1	(8)	Fa→~Ga	2,7 →I
1	(9)	(∀x)(Fx→~Gx)	8 ∀I

Proof of (3)

Problem is: ~(∀x)Fx ⊢ (∃x)~Fx

1	(1)	~(∀x)Fx	Premise
2	(2)	~(∃x)~Fx	Assumption
3	(3)	~Fa	Assumption
3	(4)	(∃x)~Fx	3 ∃I
2,3	(5)	Λ	2,4 ~E
2	(6)	~~Fa	3,5 ~I
2	(7)	Fa	6 DN
2	(8)	(∀x)Fx	7 ∀I
1,2	(9)	Λ	1,8 ~E
1	(10)	~~(∃x)~Fx	2,9 ~I
1	(11)	(∃x)~Fx	10 DN

Proof of (4)

Problem is: (∀x)(Fx→(∀y)Gy) ⊦ (∀x)(∀y)(Fx→Gy)

1	(1)	(∀x)(Fx→(∀y)Gy)	Premise
2	(2)	Fa	Assumption
1	(3)	Fa→(∀y)Gy	1 ∀E
1,2	(4)	(∀y)Gy	3,2 →E
1,2	(5)	Gb	4 ∀E
1	(6)	Fa→Gb	2,5 →I
1	(7)	(∀y)(Fa→Gy)	6 ∀I
1	(8)	(∀x)(∀y)(Fx→Gy)	7 ∀I

The Rule of ∃-Elimination: Some Background

- It is useful to think of an existential claim $\ulcorner(\exists\nu)\phi\nu\urcorner$ as a *disjunction* which asserts that the predicate expression ϕ is satisfied by *at least one* object in the domain (*i.e.*, that the disjunction $\ulcorner\phi a \vee (\phi b \vee (\phi c \vee \ldots))\urcorner$ is true).
- In this way, we would expect the elimination rule for ∃ to be similar to the elimination rule for ∨. That is, we'd expect the ∃E rule to be similar to the ∨E rule. Indeed, this is the case. It's best to start with a simple example.
- Consider the following *legitimate* elimination of an existential claim:

Problem is: (∃x)(Fx&Gx) ⊦ (∃x)Fx

1	(1)	(∃x)(Fx&Gx)	Premise
2	(2)	Fa&Ga	Assumption
2	(3)	Fa	2 &E
2	(4)	(∃x)Fx	3 ∃I
1	(5)	(∃x)Fx	1,2,4 ∃E

The Rule of ∃-Elimination: II

- To derive a sentence using the ∃E rule (with some existential sentence $\ulcorner(\exists\nu)\phi\nu\urcorner$), we must first *assume* an *instance* $\phi\tau$ of $\ulcorner(\exists\nu)\phi\nu\urcorner$.
- If we can deduce from this assumed instance $\phi\tau$ — *using **generalizable** reasoning* — then we may infer *outright*.
- It is because our reasoning from the *instance* $\phi\tau$ of $\ulcorner(\exists\nu)\phi\nu\urcorner$ to *does not depend on our choice of constant* τ (*i.e.*, that our reasoning from $\phi\tau$ to is *generalizable*) that makes this inference valid.
- When our reasoning is generalizable in this sense, it's as if we are showing that can be deduced from *any* instance $\phi\tau$ of $\ulcorner(\exists\nu)\phi\nu\urcorner$.
- As such, this is just like showing that can be deduced from *any disjunct* of the disjunction $\ulcorner\phi a \vee (\phi b \vee (\phi c \vee \ldots))\urcorner$. And, this is just like ∨E reasoning (except that ∃E only requires *one* assumption).

The Rule of ∃-Elimination: III

- Here's an *il*legitimate "∃-Elimination" step:

1	(1)	(∃x)Fx	Premise	
2	(2)	Ga	Premise	
3	(3)	Fa	Assumption	
2,3	(4)	Fa&Ga	2,3 &I	
2,3	(5)	(∃x)(Fx&Gx)	4 ∃I	
1,2	(6)	(∃x)(Fx&Gx)	1,3,5 ∃E	NO!!

- This is *not* a valid inference: $(\exists x)Fx, Ga \nvDash (\exists x)(Fx \,\&\, Gx)$!
- So, what went wrong here? The problem is that the inference to '$(\exists x)(Fx \,\&\, Gx)$' at line (5) does *not* use *generalizable* reasoning.
- We can *not* legitimately infer '$(\exists x)(Fx \,\&\, Gx)$' at line (5) from an *arbitrary instance* $\ulcorner F\tau\urcorner$ of '$(\exists x)Fx$'. We *must* assume '$F\boldsymbol{a}$' *in particular* at line (3) in order to deduce '$(\exists x)(Fx\,\&\,Gx)$' at line (5).

The Rule of ∃-Elimination: Official Definition

∃-Elimination: If ⌜$(\exists v)\phi v$⌝ occurs at i depending on $a_1, \ldots, a_n$, an instance $\phi\tau$ of ⌜$(\exists v)\phi v$⌝ is *assumed* at j, and is inferred at k depending on $b_1, \ldots, b_u$, then at line m we may infer , with label 'i, j, k ∃E' and dependencies $\{a_1, \ldots, a_n\} \cup \{b_1, \ldots, b_u\}/j$:

$a_1, \ldots, a_n$	(i)	$(\exists v)\phi v$	
	⋮		
j	(j)	$\phi\tau$	Assumption
	⋮		
$b_1, \ldots, b_u$	(k)		
	⋮		
$\{a_1, \ldots, a_n\} \cup \{b_1, \ldots, b_u\}/j$	(m)		i, j, k ∃E

Provided that ***all four*** of the following conditions are met:

- τ (in $\phi\tau$) replaces ***every*** occurrence of v in ϕv. [avoids fallacies]
- τ ***does not occur in*** ⌜$(\exists v)\phi v$⌝. [generalizability]
- τ ***does not occur in*** . [generalizability]
- τ ***does not occur in*** *any* of $b_1, \ldots, b_u$, except (possibly) $\phi\tau$ itself. [generalizability]

The Rule of ∃-Elimination: Nine Examples

- Here are 9 examples of proofs involving all four quantifier rules.

1. $(\exists x){\sim}Fx \vdash {\sim}(\forall x)Fx$ [*p*. 200, example 5]
2. $(\exists x)(Fx \to A) \vdash (\forall x)Fx \to A$ [*p*. 201, example 6]
3. $(\forall x)(\forall y)(Gy \to Fx) \vdash (\forall x)[(\exists y)Gy \to Fx]$ [*p*. 203, I. # 19 ⇒]
4. $(\exists x)[Fx \to (\forall y)Gy] \vdash (\exists x)(\forall y)(Fx \to Gy)$ [*p*. 203, I. # 20 ⇐]
5. $A \vee (\exists x)Fx \vdash (\exists x)(A \vee Fx)$ [*p*. 203, II. # 2 ⇐]
6. $(\exists x)(Fx \,\&\, {\sim}Fx) \vdash (\forall x)(Gx \,\&\, {\sim}Gx)$ [*p*. 203, I. # 12 ⇒]
7. $(\forall x)[Fx \to (\forall y){\sim}Fy] \vdash {\sim}(\exists x)Fx$ [*p*. 203, I. # 5]
8. $(\forall x)(\exists y)(Fx \,\&\, Gy) \vdash (\exists y)(\forall x)(Fx \,\&\, Gy)$ [*p*. 201, example 7]
9. $(\exists y)(\forall x)(Fx \,\&\, Gy) \vdash (\forall x)(\exists y)(Fx \,\&\, Gy)$ [other direction]

Proof of (1)

Problem is: (∃x)~Fx ⊦ ~(∀x)Fx

1	(1)	(∃x)~Fx	Premise
2	(2)	(∀x)Fx	Assumption
3	(3)	~Fa	Assumption
2	(4)	Fa	2 ∀E
2,3	(5)	Λ	3,4 ~E
1,2	(6)	Λ	1,3,5 ∃E
1	(7)	~(∀x)Fx	2,6 ~I

Proof of (2)

Problem is: (∃x)(Fx→A) ⊦ (∀x)Fx→A

1	(1)	(∃x)(Fx→A)	Premise
2	(2)	(∀x)Fx	Assumption
3	(3)	Fa→A	Assumption
2	(4)	Fa	2 ∀E
2,3	(5)	A	3,4 →E
1,2	(6)	A	1,3,5 ∃E
1	(7)	(∀x)Fx→A	2,6 →I

Proof of (3)

Problem is: (∀x)(∀y)(Gy→Fx) ⊦ (∀x)((∃y)Gy→Fx)

1	(1)	(∀x)(∀y)(Gy→Fx)	Premise
2	(2)	(∃y)Gy	Assumption
3	(3)	Gb	Assumption
1	(4)	(∀y)(Gy→Fa)	1 ∀E
1	(5)	Gb→Fa	4 ∀E
1,3	(6)	Fa	5,3 →E
1,2	(7)	Fa	2,3,6 ∃E
1	(8)	(∃y)Gy→Fa	2,7 →I
1	(9)	(∀x)((∃y)Gy→Fx)	8 ∀I

Proof of (4)

Problem is: (∃x)(Fx→(∀y)Gy) ⊦ (∃x)(∀y)(Fx→Gy)

1	(1)	(∃x)(Fx→(∀y)Gy)	Premise
2	(2)	Fa→(∀y)Gy	Assumption
3	(3)	Fa	Assumption
2,3	(4)	(∀y)Gy	2,3 →E
2,3	(5)	Gb	4 ∀E
2	(6)	Fa→Gb	3,5 →I
2	(7)	(∀y)(Fa→Gy)	6 ∀I
2	(8)	(∃x)(∀y)(Fx→Gy)	7 ∃I
1	(9)	(∃x)(∀y)(Fx→Gy)	1,2,8 ∃E

Proof of (5)

Problem is: A∨(∃x)Fx ⊦ (∃x)(A∨Fx)

1	(1)	A∨(∃x)Fx	Premise
2	(2)	A	Assumption
2	(3)	A∨Fa	2 ∨I
2	(4)	(∃x)(A∨Fx)	3 ∃I
5	(5)	(∃x)Fx	Assumption
6	(6)	Fa	Assumption
6	(7)	A∨Fa	6 ∨I
6	(8)	(∃x)(A∨Fx)	7 ∃I
5	(9)	(∃x)(A∨Fx)	5,6,8 ∃E
1	(10)	(∃x)(A∨Fx)	1,2,4,5,9 ∨E

Proof of (6)

Problem is: (∃x)(Fx&~Fx) ⊦ (∀x)(Gx&~Gx)

1	(1)	(∃x)(Fx&~Fx)	Premise
2	(2)	Fa&~Fa	Assumption
3	(3)	~Gb	Assumption
2	(4)	~Fa	2 &E
2	(5)	Fa	2 &E
2	(6)	Λ	4,5 ~E
2	(7)	~~Gb	3,6 ~I
2	(8)	Gb	7 DN
9	(9)	Gb	Assumption
2	(10)	~Gb	9,6 ~I
2	(11)	Gb&~Gb	8,10 &I
2	(12)	(∀x)(Gx&~Gx)	11 ∀I
1	(13)	(∀x)(Gx&~Gx)	1,2,12 ∃E

Proof of (7)

Problem is: (∀x)(Fx→(∀y)~Fy) ⊦ ~(∃x)Fx

1	(1)	(∀x)(Fx→(∀y)~Fy)	Premise
2	(2)	(∃x)Fx	Assumption
3	(3)	Fa	Assumption
1	(4)	Fa→(∀y)~Fy	1 ∀E
1,3	(5)	(∀y)~Fy	4,3 →E
1,3	(6)	~Fa	5 ∀E
1,3	(7)	⋀	6,3 ~E
1,2	(8)	⋀	2,3,7 ∃E
1	(9)	~(∃x)Fx	2,8 ~I

Proof of (8)

Problem is: (∀x)(∃y)(Fx&Gy) ⊦ (∃y)(∀x)(Fx&Gy)

1	(1)	(∀x)(∃y)(Fx&Gy)	Premise
1	(2)	(∃y)(Fa&Gy)	1 ∀E
3	(3)	Fa&Gb	Assumption
1	(4)	(∃y)(Fc&Gy)	1 ∀E
5	(5)	Fc&Gd	Assumption
5	(6)	Fc	5 &E
1	(7)	Fc	4,5,6 ∃E
3	(8)	Gb	3 &E
1,3	(9)	Fc&Gb	7,8 &I
1,3	(10)	(∀x)(Fx&Gb)	9 ∀I
1,3	(11)	(∃y)(∀x)(Fx&Gy)	10 ∃I
1	(12)	(∃y)(∀x)(Fx&Gy)	2,3,11 ∃E

Proof of (9)

Problem is: (∃y)(∀x)(Fx&Gy) ⊦ (∀x)(∃y)(Fx&Gy)

1	(1)	(∃y)(∀x)(Fx&Gy)	Premise
2	(2)	(∀x)(Fx&Gb)	Assumption
2	(3)	Fa&Gb	2 ∀E
2	(4)	(∃y)(Fa&Gy)	3 ∃I
1	(5)	(∃y)(Fa&Gy)	1,2,4 ∃E
1	(6)	(∀x)(∃y)(Fx&Gy)	5 ∀I