Announcements & Such

- Administrative Stuff
  - **There will be no lecture on Thursday (4/15).**
  - **HW #5 first submission is due on Thursday.**
  - **My handout "Working with LMPL Interpretations" is posted (useful for part of HW #5). I will discuss this in class today.**
  - **From now on, my office hours are: 4–6pm Tuesdays — starting today.** [This supercedes my previous planned change of office hours.]
- Today: Chapter 6 — LMPL Semantics
  - Supplementing LSL semantics with LMPL notions.
  - New definition of *interpretation* for LMPL sentences.
  - Working with LMPL interpretations.
  - Validity and Invalidity in LMPL.
  - Next: Natural Deductions in LMPL (*i.e.*, rules for the quantifiers).

Chapter 6 — Formal Semantics for LMPL

- Venn diagrams can be useful to help us figure out and visualize the conditions under which some *simple* LMPL sentences are true or false.
- But, this technique only works for sentences that have three predicates or less. If a sentence has four predicates or more, then Venn diagrams become quite difficult to draw or comprehend. [Explain this.]
- Chapter 6 provides us with a *general* semantics for LMPL. This will allow us to understand, more generally, the conditions under which *any* (*closed*!) LMPL sentence will be true or false. [Like truth-tables for LSL.]
- In Chapter 6, we will also see a precise definition of the *semantic consequence relation* ($\vDash$) for our new theory LMPL. This will allow us to determine whether LMPL *arguments* are valid or invalid (in general).
- We begin with some new terminology . . .

Formal Semantics for LMPL I: Some Terminology

- A **domain** ($\mathcal{D}$) is a nonempty (finite) set of individuals.
- The **reference of an individual constant** τ [Ref(τ)] is the object in the domain $\mathcal{D}$ to which τ refers (*e.g.*, $\ulcorner \mathrm{Ref}(\tau) = x \urcorner$ abbreviates $\ulcorner \tau$ denotes $x \urcorner$).
- The **extension of a predicate P** [Ext(**P**)] is the set of all objects in the domain which satisfy **P** (*e.g.*, if P__ : __ is at the podium, and Ref(b) = Branden, then Ext(P) = $\{b\}$). Note: extensions are always subsets of the domain $\mathcal{D}$.
- The **instances of a** (*closed*!) **quantified sentence** $\ulcorner (\mathsf{Q}v)\phi v \urcorner$ **in a domain** $\mathcal{D}$ are the sentences one gets by replacing all occurrences of v in $\ulcorner \phi v \urcorner$ with the name of each element of $\mathcal{D}$ (*e.g.*, instances of '$(\forall x)Px$' in $\mathcal{D}$ are 'Pa', 'Pb', . . . , for each individual in $\mathcal{D}$. $\therefore$ there are $|\mathcal{D}|$ instances of $\ulcorner (\mathsf{Q}v)\phi v \urcorner$ in $\mathcal{D}$).
- An **interpretation** ($\mathcal{I}$) **of an** (*closed*!) **LMPL sentence** p (**or argument** $\mathscr{A}$) is:
  (*i*) a *domain* $\mathcal{D}$,
  (*ii*) an assignment of *extensions* to any *predicate letters* in p ($\mathscr{A}$),
  (*iii*) an assignment of *references* to any *individual constants* in p ($\mathscr{A}$), and
  (*iv*) an assignment of *truth-values* to any *sentence letters* in p ($\mathscr{A}$).

Formal Semantics for LMPL II: $\top$ and $\bot$ in LMPL

- We're now in a position to give precise *truth-conditions* for each kind of (*closed*!) LMPL sentence (augmenting the truth-table definitions of LSL).
- First, the truth conditions for the (*closed*!) *atomic* sentences of LMPL:
  - An atomic sentence **P**τ is *true* ($\top$) on an interpretation $\mathcal{I}$ if the object referred to by the individual constant τ belongs to the extension of the predicate **P** (*i.e.*, if $\tau \in$ Ext(**P**)). If τ does *not* belong to the extension of the predicate **P** — that is, if $\tau \notin$ Ext(**P**) — then **P**τ is *false* ($\bot$).
- Next, the truth conditions for the (*closed*!) *quantified* sentences of LMPL:
  - A universal sentence $\ulcorner (\forall v)\phi v \urcorner$ is *true* ($\top$) *in* $\mathcal{I}$ if *all* its instances in $\mathcal{I}$ are true. If some of its instances are false (in $\mathcal{I}$), then $\ulcorner (\forall v)\phi v \urcorner$ is *false* ($\bot$).
  - An existential sentence $\ulcorner (\exists v)\phi v \urcorner$ is *true* ($\top$) *in* $\mathcal{I}$ if *some* of its instances are true in $\mathcal{I}$. If *all* its instances are false (in $\mathcal{I}$), then it's *false* ($\bot$).
- NOTE: the usual *truth-tables* for &, $\vee$, $\rightarrow$, $\leftrightarrow$, $\sim$ are still in force in LMPL!

An Example of an LMPL Interpretation

Matrix Representation: ($\mathcal{I}$) [Ignoring sentence letters.]

	F	G
α	+	−
β	−	+

- Greek letters 'α'–'σ' (*viz.*, the objects named by the *constants* 'a'–'s') are placed in the left column, alphabetically. All of the predicates in the interpretation $\mathcal{I}$ are placed across the top row, alphabetically. '+' means 'satisfies the predicate', and '−' means 'does *not* satisfy the predicate'.
- This matrix says (in addition to $\text{Ref}(a) = \alpha$, and $\text{Ref}(b) = \beta$):
  (*i*) The *domain* $\mathcal{D}$ of $\mathcal{I}$ consists of the two objects α, β (*i.e.*, $\mathcal{D} = \{\alpha, \beta\}$).
  (*ii*) The *extension* of 'F' consists of the object α (*i.e.*, $\text{Ext}(F) = \{\alpha\}$), and the *extension* of 'G' consists of the object β (*i.e.*, $\text{Ext}(G) = \{\beta\}$).
- **Quiz**: What are the truth-values — in $\mathcal{I}$ — of the following 4 sentences?

(1) $(\exists x)Fx \,\&\, (\exists x)Gx$, (2) $(\exists x)(Fx \,\&\, Gx)$, (3) $(\forall x)(Fx \vee Gx)$, (4) $(\forall x)Fx \vee (\forall x)Gx$

Validity and Invalidity of LMPL Arguments

- An argument-form $\mathscr{A}$ in LMPL is **valid** iff there is no interpretation in which all of $\mathscr{A}$'s premises are true ($\top$), but $\mathscr{A}$'s conclusion is false ($\bot$).

Example: Consider the following LMPL argument-form:

$$(\mathscr{A}_1) \quad \begin{array}{l} (\exists x)Fx \,\&\, (\exists x)Gx \\ \therefore (\exists x)(Fx \,\&\, Gx) \end{array}$$

- We have *already* proven that $\mathscr{A}_1$ is *in*valid! We just showed that — in $\mathcal{I}$ — the only premise [(1)] of $\mathscr{A}_1$ is $\top$, but the conclusion [(2)] of $\mathscr{A}_1$ is $\bot$.
- Interpretation $\mathcal{I}$ can also be used to show that the argument-form:

$$(\mathscr{A}_2) \quad \begin{array}{l} (\forall x)(Fx \vee Gx) \\ \therefore (\forall x)Fx \vee (\forall x)Gx \end{array}$$

is invalid. Its premise (3) is $\top$ in $\mathcal{I}$, but its conclusion (4) is $\bot$ in $\mathcal{I}$.

More Practice Working with LMPL Interpretations

- Consider the following LMPL interpretation:

($\mathcal{I}_1$)

	F	G	H	I	J
α	+	+	−	+	−
β	−	−	−	+	+
γ	+	−	−	−	+

- So, $\mathcal{I}_1$ is such that: $\mathcal{D} = \{\alpha, \beta, \gamma\}$, $\text{Ext}(F) = \{\alpha, \gamma\}$, $\text{Ext}(G) = \{\alpha\}$, $\text{Ext}(H) = \varnothing$ ($\varnothing$ is the *null set*), $\text{Ext}(I) = \{\alpha, \beta\}$, and $\text{Ext}(J) = \{\beta, \gamma\}$.
- What are the $\mathcal{I}$-truth-values of the following LMPL sentences?

(5) $\sim Ja$	(8) $(\forall x)[Jx \to (Gx \vee Fx)]$
(6) $Fc \to Ic$	(9) $(\exists x)Gx \to (\forall y)(Fy \vee Gy)$
(7) $(\exists x)(Jx \leftrightarrow Hx)$	(10) $(\exists y)(\forall x)[Gy \,\&\, (Jx \to (Ix \vee Fx))]$

- These are solved on page 1 of my "Working with LMPL Interpretations".

Constructing LMPL Interpretations to Prove $\nvDash$ Claims

- The notion of *semantic consequence* ($\vDash$) in LMPL is defined in the usual way. We say that $p_1, \ldots, p_n \vDash q$ in LMPL *iff* there is no LMPL interpretation on which all of $p_1, \ldots, p_n$ are true, but q is false.
- In HW #5, you are asked to prove that $p_1, \ldots, p_n \nvDash q$, for various p's and q's. This means you must *construct* (or, *find*) LMPL interpretations on which $p_1, \ldots, p_n$ are all true, but q is false.
- On page 2 of my "Working with LMPL Interpretations" handout, I have included two problems of this kind. There, I explain in detail *how I arrived at* my interpretations. This is a method you should emulate.
- On your HW's and exams, you will ***not*** need to explain *how you arrived at* your interpretations. But, you *will* need to *demonstrate* that your interpretations *really are counterexamples* (*i.e.*, that they *really are* interpretations on which $p_1, \ldots, p_n$ are all true, but q is false).

How Do We *Prove* $\models$ Claims in LMPL?

- In LSL, we had *systematic*, truth-table procedures for proving *both* negative ($\not\models$) *and* affirmative ($\models$) semantical claims.
- The method of constructing LMPL interpretations *is* a general way to establish *negative* ($\not\models$) LMPL-semantical claims.
- We will *not* be learning any systematic methods for (*directly*) establishing *affirmative* ($\models$) LMPL-semantical claims. There *are* such methods, but they are beyond the scope of this course.[a]
- In LMPL, we will rely on *natural deduction proofs* to give us an (*in*direct) method for demonstrating the *validity* of LMPL argument-forms. We'll talk about LMPL natural deductions soon.

[a] If an LMPL argument with k predicate letters is *in*valid, then there exists a *counterexample interpretation* $\mathcal{I}$ whose domain $\mathcal{D}$ has no more than 2^k elements. So, *exhaustive search* over *all* interpretations such that $|\mathcal{D}| \leq 2^k$ *is* a *decision procedure* for LMPL-validity. Note: this means checking $2^{2^k \cdot k}$ matrices. This is too many to check, even for small k. If $k = 2$, then $2^{2^k \cdot k} = 2^8 = 256$. For $k = 3$, this is 16777216! See pages 212–215 of Hunter's *Metalogic* (our 140A text). We discuss this in 140A.

Construction of LMPL Interpretations: Examples

- Here are six sample problems that require you to *construct* (or, *find*) LMPL interpretations that are *counterexamples* to $\models$ claims (the first two of these are solved on *p*. 2 of my handout on constructing LMPL interpretations):

(1) $(\forall x)(Fx \to Gx), (\forall x)(Fx \to Hx) \not\models (\forall x)(Gx \to Hx)$

(2) $(\exists x)(Fx \,\&\, Gx), (\exists x)(Fx \,\&\, Hx), (\forall x)(Gx \to \sim Hx) \not\models (\forall x)[Fx \leftrightarrow (Gx \vee Hx)]$

(3) $(\forall x)Fx \leftrightarrow (\forall x)Gx \not\models (\exists x)(Fx \leftrightarrow Gx)$[a]

(4) $(\forall x)Fx \leftrightarrow A \not\models (\forall x)(Fx \leftrightarrow A)$[b]

(5) $Fa \to (\exists x)Gx \not\models (\exists x)Fx \to (\exists x)Gx$[c]

(6) $(\exists x)(\forall y)(Fx \to Gy) \not\models (\exists y)(\forall x)(Fx \to Gy)$[d]

[a] One solution: $\mathcal{D} = \{a, b\}$, $\text{Ext}(F) = \{a\}$, $\text{Ext}(G) = \{b\}$.
[b] One solution: $\mathcal{D} = \{a, b\}$, 'A' is $\perp$, $\text{Ext}(F) = \{a\}$.
[c] One solution: $\mathcal{D} = \{a, b\}$, $\text{Ext}(F) = \{b\}$, $\text{Ext}(G) = \varnothing$.
[d] One solution: $\mathcal{D} = \{a, b\}$, $\text{Ext}(F) = \{a\}$, $\text{Ext}(G) = \varnothing$.

Construction of LMPL Interpretations: Example #1

(1) $(\forall x)(Fx \to Gx), (\forall x)(Fx \to Hx) \not\models (\forall x)(Gx \to Hx)$

- To prove (1), we need to construct (find) an interpretation $\mathcal{I}$ such that:
  (i) '$(\forall x)(Fx \to Gx)$' is true in $\mathcal{I}$.
  (ii) '$(\forall x)(Fx \to Hx)$' is true in $\mathcal{I}$.
  (iii) '$(\forall x)(Gx \to Hx)$' is false in $\mathcal{I}$.
- **Step 1**: We begin — *provisionally* — with the smallest domain $\mathcal{D} = \{a\}$.
- **Step 2**: We make sure that the object a is a *counterexample* to the conclusion '$(\forall x)(Gx \to Hx)$'. That is, we make sure that the *instance* '$Ga \to Ha$' of the conclusion is *false* on $\mathcal{I}$. So, we must have $a \in \text{Ext}(G)$, but $a \notin \text{Ext}(H)$. We can achieve this by: $\text{Ext}(G) = \{a\}$, and $\text{Ext}(H) = \varnothing$.
- **Step 3**: At the same time, we try to make *both* of the premises '$(\forall x)(Fx \to Gx)$' *and* '$(\forall x)(Fx \to Hx)$' *true* on $\mathcal{I}$.

- In this case, we can make both premises true simply by ensuring that $a \notin \text{Ext}(F)$. The simplest way to do this is to stipulate that $\text{Ext}(F) = \varnothing$ — which yields the following interpretation that does the trick:

$\mathcal{I}_{(1)}$:

	F	G	H
a	−	+	−

- We have discovered an interpretation $\mathcal{I}_{(1)}$ on which '$(\forall x)(Fx \to Gx)$' and '$(\forall x)(Fx \to Hx)$' are both true, but '$(\forall x)(Gx \to Hx)$' is false (*demonstrate this*!). Therefore, claim (1) is true.
- When you're asked to prove a claim like (1), you must do *2* things:
  - *Report* an interpretation (like $\mathcal{I}_2$) which serves as a counterexample to the validity of the LMPL argument-form, *and*
  - *Demonstrate* that your interpretation *really is* a counterexample — *i.e.*, *show* that your interpretation makes all the premises true and the conclusion false, using the methods above. You do ***not*** need to explain the process which led to the *discovery* of the interpretation.

Construction of LMPL Interpretations: Example #2

(2) $(\exists x)(Fx \,\&\, Gx)$, $(\exists x)(Fx \,\&\, Hx)$, $(\forall x)(Gx \to \sim Hx) \nvDash (\forall x)[Fx \leftrightarrow (Gx \vee Hx)]$

- We need an interpretation $\mathcal{I}$ on which '$(\exists x)(Fx \,\&\, Gx)$', '$(\exists x)(Fx \,\&\, Hx)$', and '$(\forall x)(Gx \to \sim Hx)$' are all $\top$, but '$(\forall x)[Fx \leftrightarrow (Gx \vee Hx)]$' is $\perp$.
- **Step 1**: We begin with the smallest possible domain $\mathcal{D} = \{a\}$.
- **Step 2**: We make sure that a is a *counterexample* to the conclusion '$(\forall x)[Fx \leftrightarrow (Gx \vee Hx)]$'. So, we make its *instance* '$Fa \leftrightarrow (Ga \vee Ha)$' $\perp$ on $\mathcal{I}$. One way to do this is: $a \in \text{Ext}(F)$, $a \notin \text{Ext}(G)$, and $a \notin \text{Ext}(H)$. So far, we have the following: $\text{Ext}(F) = \{a\}$, and $\text{Ext}(G) = \text{Ext}(H) = \varnothing$.
- **Step 3**: Now, we must make *all three* of the premises (*i*) '$(\exists x)(Fx \,\&\, Gx)$', (*ii*) '$(\exists x)(Fx \,\&\, Hx)$', and (*iii*) '$(\forall x)(Gx \to \sim Hx)$' $\top$ on $\mathcal{I}$. In order to make (*i*) $\top$ on $\mathcal{I}$, we must ensure that there is some object in the domain $\mathcal{D}$ which satisfies *both* 'F' *and* 'G'. But, since a must *not* satisfy both 'F' and 'G', this means we will need to *add another object* b to our domain $\mathcal{D}$.

- This new object b must be such that: $b \in \text{Ext}(F)$, and $b \in \text{Ext}(G)$. Now, we have $\text{Ext}(F) = \{a, b\}$, $\text{Ext}(G) = \{b\}$, and $\text{Ext}(H) = \varnothing$.
- All that remains is to ensure that premises (*ii*) and (*iii*) are also $\top$ on $\mathcal{I}$. In order to make (*ii*) $\top$ on $\mathcal{I}$, we'll need to make sure that there is some object in $\mathcal{D}$ which satisfies *both* 'F' *and* 'H'. We could *try* to make b satisfy *all three* 'F', 'G', *and* 'H'. But, if we were to do this, then premise (*iii*) would become *false* on $\mathcal{I}$, since its *instance* '$Gb \to \sim Hb$' would then be false on $\mathcal{I}$. Thus, we'll need to *add a third object* c to $\mathcal{D}$ such that: $c \in \text{Ext}(F)$, $c \notin \text{Ext}(G)$, and $c \in \text{Ext}(H)$ — and that does the trick:

$\mathcal{I}_{(2)}$:

	F	G	H
a	+	−	−
b	+	+	−
c	+	−	+

- We have discovered an interpretation $\mathcal{I}_{(2)}$ on which '$(\exists x)(Fx \,\&\, Gx)$', '$(\exists x)(Fx \,\&\, Hx)$', and '$(\forall x)(Gx \to \sim Hx)$' are all $\top$, but on which '$(\forall x)[Fx \leftrightarrow (Gx \vee Hx)]$' is false (*demonstrate this*!). $\therefore$ claim (2) is true.

Construction of LMPL Interpretations for $\nvDash$: Procedure

1. Begin with smallest domain possible $\mathcal{D} = \{\alpha\}$.
2. Make the conclusion of the $\nvDash$ claim false (for α).
   - That is, make the α-instance of the conclusion false.
3. Try to make all premises of the $\nvDash$ claim true (for α).
   - That is, make the α-instance of each of the premises true.
4. If you succeed, then you're done. Now *report and verify* your matrix.
5. If you fail, then add a new individual β to $\mathcal{D} = \{\alpha, \beta\}$, and continue.
6. Make the conclusion of the $\nvDash$ claim false.
   - If the conclusion is an $\forall$ claim, then it's already false.
   - If it's an $\exists$, then you must make sure its b-instance is also false.
7. Make the premises of the $\nvDash$ claim true.
   - If a premise is an $\forall$ claim, then *all* its instances must be true.
   - If it's an $\exists$ claim, only *one* of its instances needs to be true.
8. If you succeed, you're done. If not, add another (γ) to $\mathcal{D}$. Repeat ...

Using Sentential Reasoning to "Verify" LMPL $\vDash$ Claims

$$(\forall x)(\exists y)(Fx \,\&\, Gy) \dashv\vDash (\exists y)(\forall x)(Fx \,\&\, Gy)$$

- To see why, think about the truth-conditions for each side:

$$(\forall x)(\exists y)(Fx \,\&\, Gy) \approx (\exists y)(Fa \,\&\, Gy) \,\&\, (\exists y)(Fb \,\&\, Gy) \,\&\, \cdots$$
$$\approx [(Fa \,\&\, Ga) \vee (Fa \,\&\, Gb) \vee \cdots] \,\&\, [(Fb \,\&\, Ga) \vee (Fb \,\&\, Gb) \vee \cdots] \,\&\, \cdots$$
$$\approx [Fa \,\&\, (Ga \vee Gb \vee \cdots)] \,\&\, [Fb \,\&\, (Ga \vee Gb \vee \cdots)] \,\&\, \cdots$$
$$\approx (Fa \,\&\, Fb \,\&\, Fc \,\&\, \cdots) \,\&\, (Ga \vee Gb \vee Gc \vee \cdots)$$

$$(\exists y)(\forall x)(Fx \,\&\, Gy) \approx (\forall x)(Fx \,\&\, Ga) \vee (\forall x)(Fx \,\&\, Gb) \vee \cdots$$
$$\approx [(Fa \,\&\, Ga) \,\&\, (Fb \,\&\, Ga) \,\&\, \cdots] \vee [(Fa \,\&\, Gb) \,\&\, (Fb \,\&\, Gb) \,\&\, \cdots] \vee \cdots$$
$$\approx [Ga \,\&\, (Fa \,\&\, Fb \,\&\, \cdots)] \vee [Gb \,\&\, (Fa \,\&\, Fb \,\&\, \cdots)] \vee \cdots$$
$$\approx (Ga \vee Gb \vee Gc \vee \cdots) \,\&\, (Fa \,\&\, Fb \,\&\, Fc \,\&\, \cdots)$$

- $\therefore$ These two formulas are *equivalent*, since the two red formulas are $(Ga \vee Gb \vee \cdots) \,\&\, (Fa \,\&\, Fb \,\&\, \cdots) \approx (Fa \,\&\, Fb \,\&\, \cdots) \,\&\, (Ga \vee Gb \vee \cdots)$