Announcements & Such

- *Fleet Foxes*
- Administrative Stuff
  - **Take-Home Mid-Term re-subs are due Thursday.**
  - ☞ **When you turn in resubmissions, make sure that you staple them to your original homework submission.**
  - I will be discussing the grade curve for the course as soon as all of the mid-term grades are in (both the take-home and the in-class).
  - Branden will not be holding office hours this week.
- Today: Chapter 4 — Natural Deduction Proofs for LSL
  - We'll be done with the LSL-*natural deduction rules* for ⊢ this week.
  - **MacLogic** — a useful computer program for natural deduction.
    - ∗ See http://fitelson.org/maclogic.htm.

☞ **Natural deductions are the most challenging topic of the course.**

The Elimination Rule for ~

Rule of ~-Elimination: For any formula q, if ⌜$\sim q$⌝ has been inferred at a line j in a proof and q at line k (j < k or j > k) then we may infer '⅄' at line m, labeling the line 'j, k ~E' and writing on its left the numbers on the left at j and on the left at k. Schematically (with j < k):

$a_1,\ldots,a_n$	(j)	$\sim q$	
	⋮		
$b_1,\ldots,b_u$	(k)	q	
	⋮		
$a_1,\ldots,a_n,b_1,\ldots,b_u$	(m)	⅄	j, k ~E

- Note: we have *added* the symbol '⅄' to the language of LSL. It is treated as if it were an *atomic sentence* of LSL. We can now use it in compound sentences (*e.g.*, '$A \to$ ⅄', '~~⅄', *etc.*).

The Introduction Rule for ~

Rule of ~-Introduction: If '⅄' has been inferred at line k in a proof and $\{a_1,\ldots,a_n\}$ are the assumption and premise numbers '⅄' depends upon, then if p is an assumption (or premise) at line j, ⌜$\sim p$⌝ may be inferred at line m, labeling the line 'j, k ~I' and writing on its left the numbers in the set $\{a_1,\ldots,a_n\}$/j.

j	(j)	p	Assumption
	⋮		
$a_1,\ldots,a_n$	(k)	⅄	
	⋮		
$\{a_1,\ldots,a_n\}$/j	(m)	$\sim p$	j, k ~I

- ~I is used (typically *with* ~E) to deduce ⌜$\sim p$⌝ *via reductio ad absurdum*, by (*i*) *assuming* p, (*ii*) deducing '⅄', and (*iii*) *discharging* the assumption.

The Rule of Double Negation (DN)

- Negation is an odd connective in our system. It not only has an introduction rule and an elimination rule, but it also has an additional rule called the *double negation* (DN) rule.
- The DN rule says that we may infer p from ⌜$\sim\sim p$⌝. Without this DN rule, we would not be able to prove certain valid LSL argument forms — *e.g.*, $\sim(A \,\&\, \sim B)\ \therefore\ (A \to B)$.

Rule of Double Negation: For any formula p, if ⌜$\sim\sim p$⌝ has been inferred at a line j in a proof, then at line k we may infer p, labeling the line 'j ' and writing on its left the numbers to the left of j.

$a_1,\ldots,a_n$	(j)	$\sim\sim p$	
$a_1,\ldots,a_n$	(k)	p	j DN

Example Proof of a *Theorem*

- Using only the rules we have learned so far, we should be able to prove the following *theorem*: $\vdash \sim(A \,\&\, \sim A)$. Let's do this one by hand first.
- Here's a simple proof, generated using MacLogic (I'll show how):

Problem is: ⊢ ~(A&~A)

1	(1)	A&~A	Assumption (!)
1	(2)	~A	1 &E
1	(3)	A	1 &E
1	(4)	⋏	2,3 ~E
	(5)	~(A&~A)	1,4 ~I

- This proof makes use of *no premises*, and its final line has *no numbers to its left* — indicating that we have succeeded in proving '$\sim(A \,\&\, \sim A)$' from *nothing at all*. It's a *theorem* (*i.e.*, a sequent with no premises)!

The Introduction Rule for ∨ (∨I)

Rule of ∨-Introduction: For any formula p, if p has been inferred at line j, then, for any formula q, *either* $\ulcorner p \vee q \urcorner$ *or* $\ulcorner q \vee p \urcorner$ may be inferred at line k, labeling the line 'j ∨I' and writing on its left the same premise and assumption numbers as appear on the left of j.

$a_1,\ldots,a_n$	(j)	p			$a_1,\ldots,a_n$	(j)	q	
	⋮			OR		⋮		
$a_1,\ldots,a_n$	(k)	$p \vee q$	j ∨I		$a_1,\ldots,a_n$	(k)	$p \vee q$	j ∨I

- The ∨I rule is very simple an intuitive. Basically, it says that you may infer a disjunction from *either* of its disjuncts.
- The *elimination* rule (∨E) for ∨, on the other hand, is considerably more complex to state and apply. It's the hardest of our rules.

The Elimination Rule for ∨ (∨E)

- First, the idea *behind* the ∨-elimination rule.
- The following argument form is valid (easily verified *via* truth-table):

$$p \vee q$$
$$p \to r$$
$$q \to r$$
$$\therefore r$$

- This argument form is called the *constructive dilemma*. In essence, the ∨E rule reflects the constructive dilemma form of reasoning and implements it in our system of natural deduction rules.
- The ∨E rule is trickier than our other rules because it requires us to make *two* assumptions. This can make it rather complicated to keep track of all of our assumptions and premises during an ∨E proof.
- Now, the official definition of ∨E . . .

Rule of ∨-Elimination: If a disjunction $\ulcorner p \vee q \urcorner$ occurs at line g of a proof, p is assumed at line h, r is derived at line i, q is assumed at line j, and r is derived at line k, then at line m we may infer r, labeling the line 'g, h, i, j, k ∨E' and writing on its left every number on the left at line g, and at line i (except h), and at line k (except j).

$a_1,\ldots,a_n$	(g)	$p \vee q$	
	⋮		
h	(h)	p	Assumption
	⋮		
$b_1,\ldots,b_u$	(i)	r	
	⋮		
j	(j)	q	Assumption
	⋮		
$c_1,\ldots,c_w$	(k)	r	
	⋮		
$\mathscr{A}$	(m)	r	g, h, i, j, k ∨E

where $\mathscr{A}$ is the set: $\{a_1,\ldots,a_n\} \cup \{b_1,\ldots,b_u\}/h \cup \{c_1,\ldots,c_w\}/j$.

An Example Involving ∨E and DN

- Here's a proof of the sequent: $A \vee B, \sim B \vdash A$.

Problem is: A∨B, ~B ⊦ A

1	(1)	A∨B	Premise
2	(2)	~B	Premise
3	(3)	~A	Assumption (for ~I)
4	(4)	A	Assumption (for ∨E)
3,4	(5)	Λ	3,4 ~E
6	(6)	B	Assumption (for ∨E)
2,6	(7)	Λ	2,6 ~E
1,2,3	(8)	Λ	1,4,5,6,7 ∨E
1,2	(9)	~~A	3,8 ~I
1,2	(10)	A	9 DN

A Simple Example Involving ∨I and ∨E

- Here's a proof of the sequent: $A \vee B \vdash B \vee A$.

Problem is: A∨B ⊦ B∨A

1	(1)	A∨B	Premise
2	(2)	A	Assumption (∨E)
2	(3)	B∨A	2 ∨I
4	(4)	B	Assumption (∨E)
4	(5)	B∨A	4 ∨I
1	(6)	B∨A	1,2,3,4,5 ∨E

Another Example Involving ∨I and Negation

- Here's a proof of the *theorem*: $\vdash A \vee \sim A$.

Problem is: ⊦ A∨~A

1	(1)	~(A∨~A)	Assumption (~I)
2	(2)	A	Assumption (~I)
2	(3)	A∨~A	2 ∨I
1,2	(4)	Λ	1,3 ~E
1	(5)	~A	2,4 ~I
1	(6)	A∨~A	5 ∨I
1	(7)	Λ	1,6 ~E
	(8)	~~(A∨~A)	1,7 ~I
	(9)	A∨~A	8 DN

A Third Example Involving ∨E

- Here's a proof of the sequent: $A \vee B, \sim B \vdash A$.

Problem is: A∨B, ~B ⊦ A

1	(1)	A∨B	Premise
2	(2)	~B	Premise
3	(3)	~A	Assumption (for ~I)
4	(4)	A	Assumption (for ∨E)
3,4	(5)	Λ	3,4 ~E
6	(6)	B	Assumption (for ∨E)
2,6	(7)	Λ	2,6 ~E
1,2,3	(8)	Λ	1,4,5,6,7 ∨E
1,2	(9)	~~A	3,8 ~I
1,2	(10)	A	9 DN

A Fourth Example Involving ∨I and ∨E

- Here's a proof of the sequent: $A \vee (B \,\&\, C) \vdash (A \vee B) \,\&\, (A \vee C)$.

1	(1)	A∨(B&C)	Premise
2	(2)	A	Assumption (∨E)
2	(3)	A∨B	2 ∨I
2	(4)	A∨C	2 ∨I
2	(5)	(A∨B)&(A∨C)	3,4 &I
6	(6)	B&C	Assumption (∨E)
6	(7)	B	6 &E
6	(8)	A∨B	7 ∨I
6	(9)	C	6 &E
6	(10)	A∨C	9 ∨I
6	(11)	(A∨B)&(A∨C)	8,10 &I
1	(12)	(A∨B)&(A∨C)	1,2,5,6,11 ∨E

Another Example Involving ∨

- Let's do a proof of: $(A \,\&\, B) \vee (A \,\&\, C) \vdash A \,\&\, (B \vee C)$

1	(1)	(A&B)∨(A&C)	Premise
2	(2)	A&B	Ass (∨E)
2	(3)	A	2 &E
4	(4)	A&C	Ass (∨E)
4	(5)	A	4 &E
1	(6)	A	1,2,3,4,5 ∨E
2	(7)	B	2 &E
2	(8)	B∨C	7 ∨I
4	(9)	C	4 &E
4	(10)	B∨C	9 ∨I
1	(11)	B∨C	1,2,8,4,10 ∨E
1	(12)	A&(B∨C)	6,11 &I

A Final Example Involving ∨ and ∼

- Let's do a proof of: $\sim A \vee B \vdash A \to B$

Problem is : ~A∨B ⊦ A→B

1	(1)	~A∨B	Premise
2	(2)	A	Assumption (→I)
3	(3)	~A	Assumption (∨E)
4	(4)	~B	Assumption (~I)
2,3	(5)	Λ	3,2 ~E
2,3	(6)	~~B	4,5 ~I
2,3	(7)	B	6 DN
8	(8)	B	Assumption (∨E)
1,2	(9)	B	1,3,7,8,8 ∨E
1	(10)	A→B	2,9 →I

General Tips on Proof Strategy and Planning

- As a first line of attack, always try to prove your conclusion by using the introduction rule for its main connective as the main strategy.
- This will indicate what assumptions, if any, need to be made and what other formulae will need to be derived. This is "working backward".
- If these other formulae also contain connectives, then try to prove them by introducing their main connectives. Work backward, as far as possible.
- When this technique can no longer be applied, inspect your current stock of premises and assumptions to see if they have any *obvious* consequences.
- If your current premises and assumption contain a disjunction $\ulcorner r \vee s \urcorner$, see if you can prove your current goal formula p from *each* of its disjuncts r and s (using your current premises and assumptions). If you think you can, then try using ∨E to prove p. If no disjunction appears anywhere in your current of premises/assumptions, then ∨E is probably not a good strategy.
- If you have tried everything you can think of to prove your current goal p, try assuming $\ulcorner \sim p \urcorner$ and aim for $\ulcorner \sim\sim p \urcorner$ by ∼E, ∼I; then use DN.

When to Make Assumptions, and When *Not* to

- In constructing a proof, any assumptions you make must eventually be discharged, so you should only make assumptions in connection with the three rules which discharge assumptions.
- In other words, if you make an assumption p in a proof, you *must* be able to give one of the following three reasons:
  1. p is the antecedent of a conditional $\ulcorner p \to q \urcorner$ you are trying to derive using the $\to$**I** rule (then, try to prove q).
  2. You are trying to derive $\ulcorner {\sim} p \urcorner$, so you assume p with an eye toward using the ${\sim}$**I** rule (then, try to prove $\curlywedge$).
  3. p is one of the disjuncts of a disjunction $\ulcorner p \vee q \urcorner$ (*somewhere in your current stock of premises and assumptions!*) to which you will be applying $\vee$**E** (then, try to prove some r from each).
- Remember, only the three rules $\to$**I**, ${\sim}$**I**, and $\vee$**E** involve making assumptions. *No other rules can discharge assumptions.*

10 More Examples Involving $\vee$I and $\vee$E

1. $(A \,\&\, B) \vee (A \,\&\, C) \vdash A$ [p. 111, ex. 2]
2. $(A \to \curlywedge) \vee (B \to \curlywedge), B \vdash {\sim}A$ [p. 116, §4.5, ex. 11]
3. $(A \vee B) \vee C \vdash A \vee (B \vee C)$ [p. 116, ex. 19]
4. $A \vee B \vdash (A \to B) \to B$ [p. 116, ex. 10]
5. $A \,\&\, B \vdash {\sim}({\sim}A \vee {\sim}B)$ [p. 116, ex. 14 ($\vdash$)]
6. $A \vee B \vdash {\sim}({\sim}A \,\&\, {\sim}B)$ [p. 116, ex. 13]
7. ${\sim}(A \,\&\, B) \vdash {\sim}A \vee {\sim}B$ [p. 116, ex. 16 ($\dashv$)]
8. ${\sim}C \vee (A \to B) \vdash (C \,\&\, A) \to B$ [not in text]
9. $\vdash (A \to B) \vee (B \to A)$ [not in text]
10. ${\sim}(A \vee B) \vdash {\sim}A \,\&\, {\sim}B$ [not in text]

Proof of Example #1

Problem is: $(A\&B)\vee(A\&C) \vdash A$

1	(1)	$(A\&B)\vee(A\&C)$	Premise
2	(2)	$A\&B$	Assumption ($\vee$E)
2	(3)	A	2 &E
4	(4)	$A\&C$	Assumption ($\vee$E)
4	(5)	A	4 &E
1	(6)	A	1,2,3,4,5 $\vee$E

Proof of Example #2

Problem is: $(A\to\curlywedge)\vee(B\to\curlywedge), B \vdash {\sim}A$

1	(1)	$(A\to\curlywedge)\vee(B\to\curlywedge)$	Premise
2	(2)	B	Premise
3	(3)	A	Assumption (~I)
4	(4)	$A\to\curlywedge$	Assumption ($\vee$E)
3,4	(5)	$\curlywedge$	4,3 $\to$E
6	(6)	$B\to\curlywedge$	Assumption ($\vee$E)
2,6	(7)	$\curlywedge$	6,2 $\to$E
1,2,3	(8)	$\curlywedge$	1,4,5,6,7 $\vee$E
1,2	(9)	${\sim}A$	3,8 ~I

Proof of Example #3

Problem is: (A∨B)∨C ⊢ A∨(B∨C)

1	(1)	(A∨B)∨C	Premise
2	(2)	A∨B	Assumption (∨E)
3	(3)	A	Assumption (∨E)
3	(4)	A∨(B∨C)	3 ∨I
5	(5)	B	Assumption (∨E)
5	(6)	B∨C	5 ∨I
5	(7)	A∨(B∨C)	6 ∨I
2	(8)	A∨(B∨C)	2,3,4,5,7 ∨E
9	(9)	C	Assumption (∨E)
9	(10)	B∨C	9 ∨I
9	(11)	A∨(B∨C)	10 ∨I
1	(12)	A∨(B∨C)	1,2,8,9,11 ∨E

Proof of Example #4

Problem is : A∨B ⊢ (A→B)→B

1	(1)	A∨B	Premise
2	(2)	A→B	Ass (→I)
3	(3)	A	Ass (∨E)
2,3	(4)	B	2,3 →E
5	(5)	B	Ass (∨E)
1,2	(6)	B	1,3,4,5,5 ∨E
1	(7)	(A→B)→B	2,6 →I

Proof of Example #5

Problem is: A&B ⊢ ~(~A∨~B)

1	(1)	A&B	Premise
2	(2)	~A∨~B	Assumption (~I)
3	(3)	~A	Assumption (∨E)
1	(4)	A	1 &E
1,3	(5)	⋏	3,4 ~E
6	(6)	~B	Assumption (∨E)
1	(7)	B	1 &E
1,6	(8)	⋏	6,7 ~E
1,2	(9)	⋏	2,3,5,6,8 ∨E
1	(10)	~(~A∨~B)	2,9 ~I

Proof of Example #6

Problem is : A∨B ⊢ ~(~A&~B)

1	(1)	A∨B	Premise
2	(2)	~A&~B	Ass (~I)
3	(3)	A	Ass (∨E)
2	(4)	~A	2 &E
2,3	(5)	⋏	4,3 ~E
6	(6)	B	Ass (∨E)
2	(7)	~B	2 &E
2,6	(8)	⋏	7,6 ~E
1,2	(9)	⋏	1,3,5,6,8 ∨E
1	(10)	~(~A&~B)	2,9 ~I

Proof of Example #7

Problem is: ~(A&B) ⊦ ~A∨~B

1	(1)	~(A&B)	Premise
2	(2)	~(~A∨~B)	Assumption (~I)
3	(3)	~A	Assumption (~I)
3	(4)	~A∨~B	3 ∨I
2,3	(5)	⋀	2,4 ~E
2	(6)	~~A	3,5 ~I
2	(7)	A	6 DN
8	(8)	~B	Assumption (~I)
8	(9)	~A∨~B	8 ∨I
2,8	(10)	⋀	2,9 ~E
2	(11)	~~B	8,10 ~I
2	(12)	B	11 DN
2	(13)	A&B	7,12 &I
1,2	(14)	⋀	1,13 ~E
1	(15)	~~(~A∨~B)	2,14 ~I
1	(16)	~A∨~B	15 DN

Proof of Example #8

Problem is: ~C∨(A→B) ⊦ (C&A)→B

1	(1)	~C∨(A→B)	Premise
2	(2)	C&A	Assumption (→I)
3	(3)	~B	Assumption (~I)
4	(4)	~C	Assumption (∨E)
2	(5)	C	2 &E
2,4	(6)	⋀	4,5 ~E
7	(7)	A→B	Assumption (∨E)
2	(8)	A	2 &E
2,7	(9)	B	7,8 →E
2,3,7	(10)	⋀	3,9 ~E
1,2,3	(11)	⋀	1,4,6,7,10 ∨E
1,2	(12)	~~B	3,11 ~I
1,2	(13)	B	12 DN
1	(14)	(C&A)→B	2,13 →I

Proof of Example #9

Problem is: ⊢(A → B) ∨ (B → A)

1	(1)	~((A→B)∨(B→A))	Assumption (~I)
2	(2)	B	Assumption (→I)
3	(3)	~A	Assumption (~I)
4	(4)	A	Assumption (→I)
2	(5)	A→B	4,2 →I
2	(6)	(A→B)∨(B→A)	5 ∨I
1,2	(7)	⋀	1,6 ~E
1,2	(8)	~~A	3,7 ~I
1,2	(9)	A	8 DN
1	(10)	B→A	2,9 →I
1	(11)	(A→B)∨(B→A)	10 ∨I
1	(12)	⋀	1,11 ~E
	(13)	~~((A→B)∨(B→A))	1,12 ~I
	(14)	(A→B)∨(B→A)	13 DN

Proof of Example #10

Problem is : ~(A∨B) ⊦ ~A&~B

1	(1)	~(A∨B)	Premise
2	(2)	A	Ass (~I)
2	(3)	A∨B	2 ∨I
1,2	(4)	⋀	1,3 ~E
1	(5)	~A	2,4 ~I
6	(6)	B	Ass (~I)
6	(7)	A∨B	6 ∨I
1,6	(8)	⋀	1,7 ~E
1	(9)	~B	6,8 ~I
1	(10)	~A&~B	5,9 &I