Announcements & Such

- *Fleet Foxes*
- Administrative Stuff
  - **Take-Home Mid-Term re-subs are due Thursday.**
    - ☞ **When you turn in resubmissions, make sure that you staple them to your original homework submission.**
  - I will be discussing the grade curve for the course as soon as all of the mid-term grades are in (both the take-home and the in-class).
  - Branden will not be holding office hours this week.
- Today: Chapter 4 — Natural Deduction Proofs for LSL
  - We'll be done with the LSL-*natural deduction rules* for $\vdash$ this week.
  - **`MacLogic`** — a useful computer program for natural deduction.
    - ∗ See http://fitelson.org/maclogic.htm.

☞ **Natural deductions are the most challenging topic of the course.**

The Elimination Rule for ~

Rule of ~-Elimination: For any formula q, if ⌜$\sim q$⌝ has been inferred at a line j in a proof and q at line k (j < k or j > k) then we may infer '⋏' at line m, labeling the line 'j, k ~E' and writing on its left the numbers on the left at j and on the left at k. Schematically (with j < k):

$$\begin{array}{rccl} a_1,\ldots,a_n & (j) & \sim q & \\ & \vdots & & \\ b_1,\ldots,b_u & (k) & q & \\ & \vdots & & \\ a_1,\ldots,a_n, b_1,\ldots,b_u & (m) & \curlywedge & \text{j, k} \sim\text{E} \end{array}$$

- Note: we have *added* the symbol '⋏' to the language of LSL. It is treated as if it were an *atomic sentence* of LSL. We can now use it in compound sentences (*e.g.*, '$A \rightarrow \curlywedge$', '$\sim\sim\curlywedge$', *etc.*).

The Introduction Rule for $\sim$

Rule of $\sim$-Introduction: If '$\curlywedge$' has been inferred at line k in a proof and $\{a_1, \ldots, a_n\}$ are the assumption and premise numbers '$\curlywedge$' depends upon, then if p is an assumption (or premise) at line j, $\ulcorner \sim p \urcorner$ may be inferred at line m, labeling the line 'j, k $\sim$I' and writing on its left the numbers in the set $\{a_1, \ldots, a_n\}$/j.

j	(j)	p	Assumption
	$\vdots$		
$a_1, \ldots, a_n$	(k)	$\curlywedge$	
	$\vdots$		
$\{a_1, \ldots, a_n\}$/j	(m)	$\sim p$	j, k $\sim$I

- $\sim$I is used (typically *with* $\sim$E) to deduce $\ulcorner \sim p \urcorner$ *via reductio ad absurdum*, by (*i*) *assuming* p, (*ii*) deducing '$\curlywedge$', and (*iii*) *discharging* the assumption.

The Rule of Double Negation (DN)

- Negation is an odd connective in our system. It not only has an introduction rule and an elimination rule, but it also has an additional rule called the *double negation* (DN) rule.
- The DN rule says that we may infer p from $\ulcorner{\sim}{\sim}p\urcorner$. Without this DN rule, we would not be able to prove certain valid LSL argument forms — *e.g.*, ${\sim}(A \mathbin{\&} {\sim}B) \;\therefore\; (A \rightarrow B)$.

Rule of Double Negation: For any formula p, if $\ulcorner{\sim}{\sim}p\urcorner$ has been inferred at a line j in a proof, then at line k we may infer p, labeling the line 'j ' and writing on its left the numbers to the left of j.

$a_1, \ldots, a_n$	(j)	${\sim}{\sim}p$	
$a_1, \ldots, a_n$	(k)	p	j DN

Example Proof of a *Theorem*

- Using only the rules we have learned so far, we should be able to prove the following *theorem*: $\vdash \sim(A \,\&\, \sim A)$. Let's do this one by hand first.

- Here's a simple proof, generated using MacLogic (I'll show how):

Problem is: $\vdash$ ~(A&~A)

1	(1)	A&~A	Assumption (!)
1	(2)	~A	1 &E
1	(3)	A	1 &E
1	(4)	$\curlywedge$	2,3 ~E
	(5)	~(A&~A)	1,4 ~I

- This proof makes use of *no premises*, and its final line has *no numbers to its left* — indicating that we have succeeded in proving '$\sim(A \,\&\, \sim A)$' from *nothing at all*. It's a *theorem* (*i.e.*, a sequent with no premises)!

The Introduction Rule for $\vee$ ($\vee$I)

Rule of $\vee$-Introduction: For any formula p, if p has been inferred at line j, then, for any formula q, *either* $\ulcorner p \vee q \urcorner$ *or* $\ulcorner q \vee p \urcorner$ may be inferred at line k, labeling the line 'j $\vee$I' and writing on its left the same premise and assumption numbers as appear on the left of j.

$a_1, \ldots, a_n$	(j)	p			$a_1, \ldots, a_n$	(j)	q	
	$\vdots$			OR		$\vdots$		
$a_1, \ldots, a_n$	(k)	$p \vee q$	j $\vee$I		$a_1, \ldots, a_n$	(k)	$p \vee q$	j $\vee$I

- The $\vee$I rule is very simple an intuitive. Basically, it says that you may infer a disjunction from *either* of its disjuncts.
- The *elimination* rule ($\vee$E) for $\vee$, on the other hand, is considerably more complex to state and apply. It's the hardest of our rules.

The Elimination Rule for $\vee$ ($\vee$E)

- First, the idea *behind* the $\vee$-elimination rule.
- The following argument form is valid (easily verified *via* truth-table):

$$\begin{array}{l} p \vee q \\ p \to r \\ q \to r \\ \therefore r \end{array}$$

- This argument form is called the *constructive dilemma*. In essence, the $\vee$E rule reflects the constructive dilemma form of reasoning and implements it in our system of natural deduction rules.
- The $\vee$E rule is trickier than our other rules because it requires us to make *two* assumptions. This can make it rather complicated to keep track of all of our assumptions and premises during an $\vee$E proof.
- Now, the official definition of $\vee$E . . .

Rule of ∨-Elimination: If a disjunction ⌜$p \vee q$⌝ occurs at line g of a proof, p is assumed at line h, r is derived at line i, q is assumed at line j, and r is derived at line k, then at line m we may infer r, labeling the line 'g, h, i, j, k ∨E' and writing on its left every number on the left at line g, and at line i (except h), and at line k (except j).

$a_1, \ldots, a_n$	(g)	$p \vee q$	
	⋮		
h	(h)	p	Assumption
	⋮		
$b_1, \ldots, b_u$	(i)	r	
	⋮		
j	(j)	q	Assumption
	⋮		
$c_1, \ldots, c_w$	(k)	r	
	⋮		
$\mathscr{A}$	(m)	r	g, h, i, j, k ∨E

where $\mathscr{A}$ is the set: $\{a_1, \ldots, a_n\} \cup \{b_1, \ldots, b_u\}/h \cup \{c_1, \ldots, c_w\}/j$.

An Example Involving $\vee$E and DN

- Here's a proof of the sequent: $A \vee B, \sim B \vdash A$.

Problem is: $A \vee B, \sim B \vdash A$

1	(1)	$A \vee B$	Premise
2	(2)	$\sim B$	Premise
3	(3)	$\sim A$	Assumption (for $\sim$I)
4	(4)	A	Assumption (for $\vee$E)
3,4	(5)	$\wedge$	3,4 $\sim$E
6	(6)	B	Assumption (for $\vee$E)
2,6	(7)	$\wedge$	2,6 $\sim$E
1,2,3	(8)	$\wedge$	1,4,5,6,7 $\vee$E
1,2	(9)	$\sim\sim A$	3,8 $\sim$I
1,2	(10)	A	9 DN

A Simple Example Involving $\vee$I and $\vee$E

- Here's a proof of the sequent: $A \vee B \vdash B \vee A$.

Problem is: $A \vee B \vdash B \vee A$

1	(1)	$A \vee B$	Premise
2	(2)	A	Assumption ($\vee$E)
2	(3)	$B \vee A$	2 $\vee$I
4	(4)	B	Assumption ($\vee$E)
4	(5)	$B \vee A$	4 $\vee$I
1	(6)	$B \vee A$	1,2,3,4,5 $\vee$E

Another Example Involving $\vee$I and Negation

- Here's a proof of the *theorem*: $\vdash A \vee \sim A$.

Problem is: $\vdash A \vee \sim A$

1	(1)	$\sim(A \vee \sim A)$	Assumption ($\sim$I)
2	(2)	A	Assumption ($\sim$I)
2	(3)	$A \vee \sim A$	2 $\vee$I
1,2	(4)	$\wedge$	1,3 $\sim$E
1	(5)	$\sim A$	2,4 $\sim$I
1	(6)	$A \vee \sim A$	5 $\vee$I
1	(7)	$\wedge$	1,6 $\sim$E
	(8)	$\sim\sim(A \vee \sim A)$	1,7 $\sim$I
	(9)	$A \vee \sim A$	8 DN

A Third Example Involving $\vee$E

- Here's a proof of the sequent: $A \vee B, {\sim}B \vdash A$.

Problem is: $A \vee B, {\sim}B \vdash A$

1	(1)	$A \vee B$	Premise
2	(2)	${\sim}B$	Premise
3	(3)	${\sim}A$	Assumption (for ~I)
4	(4)	A	Assumption (for $\vee$E)
3,4	(5)	$\curlywedge$	3,4 ~E
6	(6)	B	Assumption (for $\vee$E)
2,6	(7)	$\curlywedge$	2,6 ~E
1,2,3	(8)	$\curlywedge$	1,4,5,6,7 $\vee$E
1,2	(9)	${\sim}{\sim}A$	3,8 ~I
1,2	(10)	A	9 DN

A Fourth Example Involving ∨I and ∨E

- Here's a proof of the sequent: $A \vee (B \,\&\, C) \vdash (A \vee B) \,\&\, (A \vee C)$.

1	(1)	A∨(B&C)	Premise
2	(2)	A	Assumption (∨E)
2	(3)	A∨B	2 ∨I
2	(4)	A∨C	2 ∨I
2	(5)	(A∨B)&(A∨C)	3,4 &I
6	(6)	B&C	Assumption (∨E)
6	(7)	B	6 &E
6	(8)	A∨B	7 ∨I
6	(9)	C	6 &E
6	(10)	A∨C	9 ∨I
6	(11)	(A∨B)&(A∨C)	8,10 &I
1	(12)	(A∨B)&(A∨C)	1,2,5,6,11 ∨E

Another Example Involving $\vee$

- Let's do a proof of: $(A \,\&\, B) \vee (A \,\&\, C) \vdash A \,\&\, (B \vee C)$

1	(1)	(A&B)∨(A&C)	Premise
2	(2)	A&B	Ass (∨E)
2	(3)	A	2 &E
4	(4)	A&C	Ass (∨E)
4	(5)	A	4 &E
1	(6)	A	1,2,3,4,5 ∨E
2	(7)	B	2 &E
2	(8)	B∨C	7 ∨I
4	(9)	C	4 &E
4	(10)	B∨C	9 ∨I
1	(11)	B∨C	1,2,8,4,10 ∨E
1	(12)	A&(B∨C)	6,11 &I

A Final Example Involving $\vee$ and $\sim$

- Let's do a proof of: $\sim A \vee B \vdash A \to B$

Problem is : $\sim A \vee B \vdash A \to B$

1	(1)	$\sim A \vee B$	Premise
2	(2)	A	Assumption ($\to$I)
3	(3)	$\sim A$	Assumption ($\vee$E)
4	(4)	$\sim B$	Assumption ($\sim$I)
2,3	(5)	$\wedge$	3,2 $\sim$E
2,3	(6)	$\sim\sim B$	4,5 $\sim$I
2,3	(7)	B	6 DN
8	(8)	B	Assumption ($\vee$E)
1,2	(9)	B	1,3,7,8,8 $\vee$E
1	(10)	$A \to B$	2,9 $\to$I

General Tips on Proof Strategy and Planning

- As a first line of attack, always try to prove your conclusion by using the introduction rule for its main connective as the main strategy.
- This will indicate what assumptions, if any, need to be made and what other formulae will need to be derived. This is "working backward".
- If these other formulae also contain connectives, then try to prove them by introducing their main connectives. Work backward, as far as possible.
- When this technique can no longer be applied, inspect your current stock of premises and assumptions to see if they have any *obvious* consequences.
- If your current premises and assumption contain a disjunction $\ulcorner r \vee s \urcorner$, see if you can prove your current goal formula p from *each* of its disjuncts r and s (using your current premises and assumptions). If you think you can, then try using $\vee$E to prove p. If no disjunction appears anywhere in your current of premises/assumptions, then $\vee$E is probably not a good strategy.
- If you have tried everything you can think of to prove your current goal p, try assuming $\ulcorner {\sim}p \urcorner$ and aim for $\ulcorner {\sim}{\sim}p \urcorner$ by ${\sim}$E, ${\sim}$I; then use DN.

When to Make Assumptions, and When *Not* to

- In constructing a proof, any assumptions you make must eventually be discharged, so you should only make assumptions in connection with the three rules which discharge assumptions.
- In other words, if you make an assumption p in a proof, you *must* be able to give one of the following three reasons:
  1. p is the antecedent of a conditional $\ulcorner p \to q \urcorner$ you are trying to derive using the $\to$**I** rule (then, try to prove q).
  2. You are trying to derive $\ulcorner \sim p \urcorner$, so you assume p with an eye toward using the $\sim$**I** rule (then, try to prove $\curlywedge$).
  3. p is one of the disjuncts of a disjunction $\ulcorner p \vee q \urcorner$ (*somewhere in your current stock of premises and assumptions!*) to which you will be applying $\vee$**E** (then, try to prove some r from each).
- Remember, only the three rules $\to$**I**, $\sim$**I**, and $\vee$**E** involve making assumptions. *No other rules can discharge assumptions.*

10 More Examples Involving $\vee$I and $\vee$E

1. $(A \,\&\, B) \vee (A \,\&\, C) \vdash A$ [p. 111, ex. 2]
2. $(A \to \curlywedge) \vee (B \to \curlywedge), B \vdash \sim A$ [p. 116, §4.5, ex. 11]
3. $(A \vee B) \vee C \vdash A \vee (B \vee C)$ [p. 116, ex. 19]
4. $A \vee B \vdash (A \to B) \to B$ [p. 116, ex. 10]
5. $A \,\&\, B \vdash \sim(\sim A \vee \sim B)$ [p. 116, ex. 14 ($\vdash$)]
6. $A \vee B \vdash \sim(\sim A \,\&\, \sim B)$ [p. 116, ex. 13]
7. $\sim(A \,\&\, B) \vdash \sim A \vee \sim B$ [p. 116, ex. 16 ($\dashv$)]
8. $\sim C \vee (A \to B) \vdash (C \,\&\, A) \to B$ [not in text]
9. $\vdash (A \to B) \vee (B \to A)$ [not in text]
10. $\sim(A \vee B) \vdash \sim A \,\&\, \sim B$ [not in text]

Proof of Example #1

Problem is: $(A\&B) \vee (A\&C) \vdash A$

1	(1)	$(A\&B) \vee (A\&C)$	Premise
2	(2)	$A\&B$	Assumption ($\vee$E)
2	(3)	A	2 &E
4	(4)	$A\&C$	Assumption ($\vee$E)
4	(5)	A	4 &E
1	(6)	A	1,2,3,4,5 $\vee$E

Proof of Example #2

Problem is: $(A \to \curlywedge) \vee (B \to \curlywedge)$, $B \vdash \sim A$

1	(1)	$(A \to \curlywedge) \vee (B \to \curlywedge)$	Premise
2	(2)	B	Premise
3	(3)	A	Assumption ($\sim$I)
4	(4)	$A \to \curlywedge$	Assumption ($\vee$E)
3,4	(5)	$\curlywedge$	4,3 $\to$E
6	(6)	$B \to \curlywedge$	Assumption ($\vee$E)
2,6	(7)	$\curlywedge$	6,2 $\to$E
1,2,3	(8)	$\curlywedge$	1,4,5,6,7 $\vee$E
1,2	(9)	$\sim A$	3,8 $\sim$I

Proof of Example #3

Problem is: $(A \vee B) \vee C \vdash A \vee (B \vee C)$

1	(1)	$(A \vee B) \vee C$	Premise
2	(2)	$A \vee B$	Assumption ($\vee$E)
3	(3)	A	Assumption ($\vee$E)
3	(4)	$A \vee (B \vee C)$	3 $\vee$I
5	(5)	B	Assumption ($\vee$E)
5	(6)	$B \vee C$	5 $\vee$I
5	(7)	$A \vee (B \vee C)$	6 $\vee$I
2	(8)	$A \vee (B \vee C)$	2,3,4,5,7 $\vee$E
9	(9)	C	Assumption ($\vee$E)
9	(10)	$B \vee C$	9 $\vee$I
9	(11)	$A \vee (B \vee C)$	10 $\vee$I
1	(12)	$A \vee (B \vee C)$	1,2,8,9,11 $\vee$E

Proof of Example #4

Problem is : $A \vee B \vdash (A \to B) \to B$

1	(1)	$A \vee B$	Premise
2	(2)	$A \to B$	Ass ($\to$I)
3	(3)	A	Ass ($\vee$E)
2,3	(4)	B	2,3 $\to$E
5	(5)	B	Ass ($\vee$E)
1,2	(6)	B	1,3,4,5,5 $\vee$E
1	(7)	$(A \to B) \to B$	2,6 $\to$I

Proof of Example #5

Problem is: A&B ⊢ ~(~A∨~B)

1	(1)	A&B	Premise
2	(2)	~A∨~B	Assumption (~I)
3	(3)	~A	Assumption (∨E)
1	(4)	A	1 &E
1,3	(5)	⋏	3,4 ~E
6	(6)	~B	Assumption (∨E)
1	(7)	B	1 &E
1,6	(8)	⋏	6,7 ~E
1,2	(9)	⋏	2,3,5,6,8 ∨E
1	(10)	~(~A∨~B)	2,9 ~I

Proof of Example #6

Problem is : $A \vee B \vdash \sim(\sim A \& \sim B)$

1	(1)	$A \vee B$	Premise
2	(2)	$\sim A \& \sim B$	Ass ($\sim$I)
3	(3)	A	Ass ($\vee$E)
2	(4)	$\sim A$	2 &E
2,3	(5)	$\curlywedge$	4,3 $\sim$E
6	(6)	B	Ass ($\vee$E)
2	(7)	$\sim B$	2 &E
2,6	(8)	$\curlywedge$	7,6 $\sim$E
1,2	(9)	$\curlywedge$	1,3,5,6,8 $\vee$E
1	(10)	$\sim(\sim A \& \sim B)$	2,9 $\sim$I

Proof of Example #7

Problem is: ~(A&B) ⊢ ~A∨~B

1	(1)	~(A&B)	Premise
2	(2)	~(~A∨~B)	Assumption (~I)
3	(3)	~A	Assumption (~I)
3	(4)	~A∨~B	3 ∨I
2,3	(5)	⋏	2,4 ~E
2	(6)	~~A	3,5 ~I
2	(7)	A	6 DN
8	(8)	~B	Assumption (~I)
8	(9)	~A∨~B	8 ∨I
2,8	(10)	⋏	2,9 ~E
2	(11)	~~B	8,10 ~I
2	(12)	B	11 DN
2	(13)	A&B	7,12 &I
1,2	(14)	⋏	1,13 ~E
1	(15)	~~(~A∨~B)	2,14 ~I
1	(16)	~A∨~B	15 DN

Proof of Example #8

Problem is: $\sim C \vee (A \to B) \vdash (C \& A) \to B$

1	(1)	$\sim C \vee (A \to B)$	Premise
2	(2)	$C \& A$	Assumption ($\to$I)
3	(3)	$\sim B$	Assumption ($\sim$I)
4	(4)	$\sim C$	Assumption ($\vee$E)
2	(5)	C	2 &E
2,4	(6)	$\curlywedge$	4,5 $\sim$E
7	(7)	$A \to B$	Assumption ($\vee$E)
2	(8)	A	2 &E
2,7	(9)	B	7,8 $\to$E
2,3,7	(10)	$\curlywedge$	3,9 $\sim$E
1,2,3	(11)	$\curlywedge$	1,4,6,7,10 $\vee$E
1,2	(12)	$\sim\sim B$	3,11 $\sim$I
1,2	(13)	B	12 DN
1	(14)	$(C \& A) \to B$	2,13 $\to$I

Proof of Example #9

Problem is: $\vdash (A \to B) \vee (B \to A)$

1	(1)	$\sim((A \to B) \vee (B \to A))$	Assumption ($\sim$I)
2	(2)	B	Assumption ($\to$I)
3	(3)	$\sim A$	Assumption ($\sim$I)
4	(4)	A	Assumption ($\to$I)
2	(5)	$A \to B$	4,2 $\to$I
2	(6)	$(A \to B) \vee (B \to A)$	5 $\vee$I
1,2	(7)	$\curlywedge$	1,6 $\sim$E
1,2	(8)	$\sim\sim A$	3,7 $\sim$I
1,2	(9)	A	8 DN
1	(10)	$B \to A$	2,9 $\to$I
1	(11)	$(A \to B) \vee (B \to A)$	10 $\vee$I
1	(12)	$\curlywedge$	1,11 $\sim$E
	(13)	$\sim\sim((A \to B) \vee (B \to A))$	1,12 $\sim$I
	(14)	$(A \to B) \vee (B \to A)$	13 DN

Proof of Example #10

Problem is : $\sim(A \vee B) \vdash \sim A \& \sim B$

1	(1)	$\sim(A \vee B)$	Premise
2	(2)	A	Ass ($\sim$I)
2	(3)	$A \vee B$	2 $\vee$I
1,2	(4)	$\curlywedge$	1,3 $\sim$E
1	(5)	$\sim A$	2,4 $\sim$I
6	(6)	B	Ass ($\sim$I)
6	(7)	$A \vee B$	6 $\vee$I
1,6	(8)	$\curlywedge$	1,7 $\sim$E
1	(9)	$\sim B$	6,8 $\sim$I
1	(10)	$\sim A \& \sim B$	5,9 &I