Announcements & Such

• Fleet Foxes

• Administrative Stuff
  – Take-Home Mid-Term re-subs are due Thursday.
    ⚠️ When you turn in resubmissions, make sure that you staple them to your original homework submission.
  – I will be discussing the grade curve for the course as soon as all of the mid-term grades are in (both the take-home and the in-class).
  – Branden will not be holding office hours this week.

• Today: Chapter 4 — Natural Deduction Proofs for LSL
  – We’ll be done with the LSL-natural deduction rules for ⊢ this week.
  – MacLogic — a useful computer program for natural deduction.
  ⚠️ Natural deductions are the most challenging topic of the course.
The Elimination Rule for $\sim$

**Rule of $\sim$-Elimination:** For any formula $q$, if $\sim q$ has been inferred at a line $j$ in a proof and $q$ at line $k$ ($j < k$ or $j > k$) then we may infer ‘$\land$’ at line $m$, labeling the line ‘$j, k \sim E$’ and writing on its left the numbers on the left at $j$ and on the left at $k$. Schematically (with $j < k$):

$$
a_1, \ldots, a_n \quad (j) \quad \sim q \\
\vdots \\
b_1, \ldots, b_u \quad (k) \quad q \\
\vdots \\
a_1, \ldots, a_n, b_1, \ldots, b_u \quad (m) \quad \land \quad j, k \sim E
$$

• Note: we have *added* the symbol ‘$\land$’ to the language of LSL. It is treated as if it were an *atomic sentence* of LSL. We can now use it in compound sentences (*e.g.*, ‘$A \rightarrow \land$’, ‘$\sim \sim \land$’, *etc.*).
The Introduction Rule for \(\sim\)

**Rule of \(\sim\)-Introduction:** If ‘\(\land\)’ has been inferred at line \(k\) in a proof and \(\{a_1, \ldots, a_n\}\) are the assumption and premise numbers ‘\(\land\)’ depends upon, then if \(p\) is an assumption (or premise) at line \(j\), \(\sim p\) may be inferred at line \(m\), labeling the line ‘\(j, k \sim I\)’ and writing on its left the numbers in the set \(\{a_1, \ldots, a_n\}/j\).

\[
\begin{array}{c}
\text{j} & (j) & p & \text{Assumption} \\
\vdots \\
a_1, \ldots, a_n & (k) & \land & \\
\vdots \\
\{a_1, \ldots, a_n\}/j & (m) & \sim p & j, k \sim I \\
\end{array}
\]

- \(\sim I\) is used (typically with \(\sim E\)) to deduce \(\sim p\) *via reductio ad absurdum*, by (i) assuming \(p\), (ii) deducing ‘\(\land\)’, and (iii) discharging the assumption.
The Rule of Double Negation (DN)

- Negation is an odd connective in our system. It not only has an introduction rule and an elimination rule, but it also has an additional rule called the *double negation* (DN) rule.
- The DN rule says that we may infer \( p \) from \( \sim \sim p \). Without this DN rule, we would not be able to prove certain valid LSL argument forms — *e.g.*, \( \sim (A \& \sim B) \therefore (A \rightarrow B) \).

**Rule of Double Negation:** For any formula \( p \), if \( \sim \sim p \) has been inferred at a line \( j \) in a proof, then at line \( k \) we may infer \( p \), labeling the line ‘j’ and writing on its left the numbers to the left of \( j \).

\[
\begin{align*}
\text{a}_1, \ldots, \text{a}_n & \quad (j) \quad \sim \sim p \\
\text{a}_1, \ldots, \text{a}_n & \quad (k) \quad p \quad j \text{ DN}
\end{align*}
\]
Example Proof of a *Theorem*

- Using only the rules we have learned so far, we should be able to prove the following *theorem*: \( \vdash \sim (A \& \sim A) \). Let’s do this one by hand first.

- Here’s a simple proof, generated using MacLogic (I’ll show how):

  **Problem is:**  \( \vdash \sim (A \& \sim A) \)

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  - This proof makes use of *no premises*, and its final line has *no numbers to its left* — indicating that we have succeeded in proving ‘\( \sim (A \& \sim A) \)’ from *nothing at all*. It’s a *theorem* (i.e., a sequent with no premises)!
The Introduction Rule for \( \lor \) (\( \lor \text{I} \))

**Rule of \( \lor \)-Introduction:** For any formula \( p \), if \( p \) has been inferred at line \( j \), then, for any formula \( q \), either \( \lnot \lnot p \lor q \lor q \lor p \) may be inferred at line \( k \), labeling the line ‘\( j \lor \text{I} \)’ and writing on its left the same premise and assumption numbers as appear on the left of \( j \).

\[
\begin{align*}
\text{a}_1, \ldots, \text{a}_n \quad (j) & \quad p \\
\vdots & \quad \text{OR} \\
\text{a}_1, \ldots, \text{a}_n \quad (k) & \quad p \lor q \quad j \lor \text{I} \\
\end{align*}
\]

- The \( \lor \text{I} \) rule is very simple and intuitive. Basically, it says that you may infer a disjunction from *either* of its disjuncts.

- The *elimination* rule (\( \lor \text{E} \)) for \( \lor \), on the other hand, is considerably more complex to state and apply. It’s the hardest of our rules.
The Elimination Rule for $\lor$ ($\lor$E)

- First, the idea *behind* the $\lor$-elimination rule.
- The following argument form is valid (easily verified *via* truth-table):
  
  $$
  p \lor q \\
  p \rightarrow r \\
  q \rightarrow r \\
  \therefore r
  $$

- This argument form is called the *constructive dilemma*. In essence, the $\lor$E rule reflects the constructive dilemma form of reasoning and implements it in our system of natural deduction rules.
- The $\lor$E rule is trickier than our other rules because it requires us to make *two* assumptions. This can make it rather complicated to keep track of all of our assumptions and premises during an $\lor$E proof.
- Now, the official definition of $\lor$E ...
**Rule of ∨-Elimination:** If a disjunction \( p ∨ q \) occurs at line \( g \) of a proof, \( p \) is assumed at line \( h \), \( r \) is derived at line \( i \), \( q \) is assumed at line \( j \), and \( r \) is derived at line \( k \), then at line \( m \) we may infer \( r \), labeling the line ‘g, h, i, j, k ∨E’ and writing on its left every number on the left at line \( g \), and at line \( i \) (except \( h \)), and at line \( k \) (except \( j \)).

\[
\begin{align*}
a_1, \ldots, a_n & \quad \text{(g)} \quad p ∨ q \\
& \quad \vdots \\
& \quad h & \quad \text{(h)} \quad p \quad \text{Assumption} \\
& \quad \vdots \\
& b_1, \ldots, b_u & \quad \text{(i)} \quad r \\
& \quad \vdots \\
& j & \quad \text{(j)} \quad q \quad \text{Assumption} \\
& \quad \vdots \\
& c_1, \ldots, c_w & \quad \text{(k)} \quad r \\
& \quad \vdots \\
& \mathcal{A} & \quad \text{(m)} \quad r \quad g, h, i, j, k ∨E
\end{align*}
\]

where \( \mathcal{A} \) is the set: \( \{a_1, \ldots, a_n\} \cup \{b_1, \ldots, b_u\}/h \cup \{c_1, \ldots, c_w\}/j \).
An Example Involving \( \lor E \) and DN

Here’s a proof of the sequent: \( A \lor B, \neg B \vdash A \).

Problem is: \( A \lor B, \neg B \vdash A \)

1. (1) \( A \lor B \)  
   Premise
2. (2) \( \neg B \)  
   Premise
3. (3) \( \neg A \)  
   Assumption (for \( \neg I \))
4. (4) \( A \)  
   Assumption (for \( \lor E \))
3, 4  
5. (5) \( \neg \)  
   \( 3, 4 \ \neg E \)
6. (6) \( B \)  
   Assumption (for \( \lor E \))
2, 6  
7. (7) \( \neg \)  
   \( 2, 6 \ \neg E \)
1, 2, 3  
8. (8) \( \neg \)  
   \( 1, 4, 5, 6, 7 \ \lor E \)
1, 2  
9. (9) \( \neg \neg A \)  
   \( 3, 8 \ \neg I \)
1, 2  
10. (10) \( A \)  
   \( 9 \ \text{DN} \)
A Simple Example Involving \( \lor I \) and \( \lor E \)

Here’s a proof of the sequent: \( A \lor B \vdash B \lor A \).

Problem is: \( A \lor B \vdash B \lor A \)

\[
\begin{align*}
1 & \quad (1) \quad A \lor B & \text{Premise} \\
2 & \quad (2) \quad A & \text{Assumption (}\lor E\text{)} \\
2 & \quad (3) \quad B \lor A & 2 \lor I \\
4 & \quad (4) \quad B & \text{Assumption (}\lor E\text{)} \\
4 & \quad (5) \quad B \lor A & 4 \lor I \\
1 & \quad (6) \quad B \lor A & 1,2,3,4,5 \lor E
\end{align*}
\]
Another Example Involving $\lor$ I and Negation

Here’s a proof of the theorem: $\vdash A \lor \neg A$.

Problem is: $\vdash A \lor \neg A$

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<td>$A \lor \neg A$</td>
<td>8 DN</td>
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A Third Example Involving $\lor E$

- Here’s a proof of the sequent: $A \lor B, \sim B \vdash A$.

Problem is: $A \lor B, \sim B \vdash A$

\begin{align*}
1 & \quad (1) \quad A \lor B \quad \text{Premise} \\
2 & \quad (2) \quad \sim B \quad \text{Premise} \\
3 & \quad (3) \quad \sim A \quad \text{Assumption (for $\sim I$)} \\
4 & \quad (4) \quad A \quad \text{Assumption (for $\lor E$)} \\
3,4 & \quad (5) \quad \Lambda \quad 3,4 \ \sim E \\
6 & \quad (6) \quad B \quad \text{Assumption (for $\lor E$)} \\
2,6 & \quad (7) \quad \Lambda \quad 2,6 \ \sim E \\
1,2,3 & \quad (8) \quad \Lambda \quad 1,4,5,6,7 \ \lor E \\
1,2 & \quad (9) \quad \sim \sim A \quad 3,8 \ \sim I \\
1,2 & \quad (10) \quad A \quad 9 \ \text{DN}
\end{align*}
A Fourth Example Involving ∨I and ∨E

- Here’s a proof of the sequent: \( A \lor (B \land C) \vdash (A \lor B) \land (A \lor C) \).

\[
\begin{array}{ll}
1 & (1) \ A \lor (B \land C) \quad \text{Premise} \\
2 & (2) \ A \quad \text{Assumption (∨E)} \\
2 & (3) \ A \lor B \\
2 & (4) \ A \lor C \\
2 & (5) \ (A \lor B) \land (A \lor C) \\
6 & (6) \ B \land C \quad \text{Assumption (∨E)} \\
6 & (7) \ B \\
6 & (8) \ A \lor B \\
6 & (9) \ C \\
6 & (10) \ A \lor C \\
6 & (11) \ (A \lor B) \land (A \lor C) \\
1 & (12) \ (A \lor B) \land (A \lor C) \\
\end{array}
\]
Another Example Involving $\lor$

Let’s do a proof of: $(A \& B) \lor (A \& C) \vdash A \& (B \lor C)$

1. (1) $(A \& B) \lor (A \& C)$ \hspace{1cm} Premise
2. (2) $A \& B$ \hspace{1cm} Ass ($\lor$E)
2. (3) $A$ \hspace{1cm} 2 \&E
4. (4) $A \& C$ \hspace{1cm} Ass ($\lor$E)
4. (5) $A$ \hspace{1cm} 4 \&E
1. (6) $A$ \hspace{1cm} 1,2,3,4,5 $\lor$E
2. (7) $B$ \hspace{1cm} 2 \&E
2. (8) $B \lor C$ \hspace{1cm} 7 $\lor$I
4. (9) $C$ \hspace{1cm} 4 \&E
4. (10) $B \lor C$ \hspace{1cm} 9 $\lor$I
1. (11) $B \lor C$ \hspace{1cm} 1,2,8,4,10 $\lor$E
1. (12) $A \& (B \lor C)$ \hspace{1cm} 6,11 \&I
A Final Example Involving $\lor$ and $\neg$

Let’s do a proof of: $\neg A \lor B \vdash A \rightarrow B$

Problem is: $\neg A \lor B \vdash A \rightarrow B$

1. (1) $\neg A \lor B$ Premise
2. (2) $A$ Assumption ($\rightarrow$I)
3. (3) $\neg A$ Assumption ($\lor$E)
4. (4) $\neg B$ Assumption ($\neg$I)
2,3 (5) $\neg \neg \neg B$ 3,2 $\neg$E
2,3 (6) $\neg \neg \neg B$ 4,5 $\neg$I
2,3 (7) $B$ 6 DN
8 (8) $B$ Assumption ($\lor$E)
1,2 (9) $B$ 1,3,7,8,8 $\lor$E
1 (10) $A \rightarrow B$ 2,9 $\rightarrow$I
General Tips on Proof Strategy and Planning

- As a first line of attack, always try to prove your conclusion by using the introduction rule for its main connective as the main strategy.
- This will indicate what assumptions, if any, need to be made and what other formulae will need to be derived. This is "working backward".
- If these other formulae also contain connectives, then try to prove them by introducing their main connectives. Work backward, as far as possible.
- When this technique can no longer be applied, inspect your current stock of premises and assumptions to see if they have any obvious consequences.
- If your current premises and assumption contain a disjunction \( r \lor s \), see if you can prove your current goal formula \( p \) from each of its disjuncts \( r \) and \( s \) (using your current premises and assumptions). If you think you can, then try using \( \lor E \) to prove \( p \). If no disjunction appears anywhere in your current of premises/assumptions, then \( \lor E \) is probably not a good strategy.
- If you have tried everything you can think of to prove your current goal \( p \), try assuming \( \sim p \) and aim for \( \sim \sim p \) by \( \sim E, \sim I \); then use DN.
When to Make Assumptions, and When Not to

- In constructing a proof, any assumptions you make must eventually be discharged, so you should only make assumptions in connection with the three rules which discharge assumptions.

- In other words, if you make an assumption $p$ in a proof, you must be able to give one of the following three reasons:
  1. $p$ is the antecedent of a conditional "$p \rightarrow q" you are trying to derive using the $\rightarrow$I rule (then, try to prove $q$).
  2. You are trying to derive "$\sim p"$, so you assume $p$ with an eye toward using the $\sim$I rule (then, try to prove $\land$).
  3. $p$ is one of the disjuncts of a disjunction "$p \lor q" (somewhere in your current stock of premises and assumptions!) to which you will be applying $\lor$E (then, try to prove some $r$ from each).

- Remember, only the three rules $\rightarrow$I, $\sim$I, and $\lor$E involve making assumptions. *No other rules can discharge assumptions.*
10 More Examples Involving \( \lor \text{I} \) and \( \lor \text{E} \)

1. \((A \land B) \lor (A \land C) \vdash A\)  
   \[p. 111, \text{ex. 2}\]

2. \((A \rightarrow \land) \lor (B \rightarrow \land), B \vdash \neg A\)  
   \[p. 116, \S 4.5, \text{ex. 11}\]

3. \((A \lor B) \lor C \vdash A \lor (B \lor C)\)  
   \[p. 116, \text{ex. 19}\]

4. \(A \lor B \vdash (A \rightarrow B) \rightarrow B\)  
   \[p. 116, \text{ex. 10}\]

5. \(A \land B \vdash \neg(\neg A \lor \neg B)\)  
   \[p. 116, \text{ex. 14 (\neg)}\]

6. \(A \lor B \vdash \neg(\neg A \land \neg B)\)  
   \[p. 116, \text{ex. 13}\]

7. \(\neg(A \land B) \vdash \neg A \lor \neg B\)  
   \[p. 116, \text{ex. 16 (\neg\rightarrow)}\]

8. \(\neg C \land (A \rightarrow B) \vdash (C \land A) \rightarrow B\)  
   \[\text{not in text}\]

9. \(\vdash (A \rightarrow B) \lor (B \rightarrow A)\)  
   \[\text{not in text}\]

10. \(\neg(A \lor B) \vdash \neg A \land \neg B\)  
    \[\text{not in text}\]
Proof of Example #1

Problem is: \((A \& B) \lor (A \& C) \vdash A\)

1. \((A \& B) \lor (A \& C)\)  \hspace{1cm} \text{Premise}
2. \((A \& B)\)  \hspace{1cm} \text{Assumption} (\lor E)
3. \(A\)  \hspace{1cm} \text{2 \& E}
4. \((A \& C)\)  \hspace{1cm} \text{Assumption} (\lor E)
5. \(A\)  \hspace{1cm} \text{4 \& E}
6. \(A\)  \hspace{1cm} 1,2,3,4,5 \lor E
Proof of Example #2

Problem is: \((A \rightarrow \Lambda) \lor (B \rightarrow \Lambda), B \vdash \neg A\)

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<td>(9) (\neg A) \hspace{1cm} 3, 8 (\neg I)</td>
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### Proof of Example #3

Problem is: \((A \lor B) \lor C \vdash A \lor (B \lor C)\)

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<td>(8)</td>
<td>(A \lor (B \lor C))</td>
<td></td>
<td>(2,3,4,5,7 \ \lor E)</td>
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<tr>
<td>9</td>
<td></td>
<td>(9)</td>
<td>(C)</td>
<td></td>
<td>Assumption ((\lor E))</td>
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<tr>
<td>9</td>
<td></td>
<td>(10)</td>
<td>(B \lor C)</td>
<td></td>
<td>(9 \ \lor I)</td>
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<tr>
<td>9</td>
<td></td>
<td>(11)</td>
<td>(A \lor (B \lor C))</td>
<td></td>
<td>(10 \ \lor I)</td>
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<tr>
<td>1</td>
<td></td>
<td>(12)</td>
<td>(A \lor (B \lor C))</td>
<td></td>
<td>(1,2,8,9,11 \ \lor E)</td>
<td></td>
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</tr>
</tbody>
</table>
Proof of Example #4

Problem is:  \( A \lor B \vdash (A \rightarrow B) \rightarrow B \)

1. (1) \( A \lor B \)  
   Premise
2. (2) \( A \rightarrow B \)  
   Ass (\( \rightarrow I \))
3. (3) \( A \)  
   Ass (\( \lor E \))
4. (4) \( B \)  
   2,3 \( \rightarrow E \)
5. (5) \( B \)  
   Ass (\( \lor E \))
6. (6) \( B \)  
   1,3,4,5,5 \( \lor E \)
7. (7) \((A \rightarrow B) \rightarrow B \)  
   2,6 \( \rightarrow I \)
### Proof of Example #5

Problem is: \(A\&B \vdash \neg(\neg A \lor \neg B)\)

<table>
<thead>
<tr>
<th>Step</th>
<th>Line</th>
<th>Formula</th>
<th>Justification</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>(A&amp;B)</td>
<td>Premise</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>(\neg A \lor \neg B)</td>
<td>Assumption ((\neg I))</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>(\neg A)</td>
<td>Assumption ((\lor E))</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>(A)</td>
<td>1 &amp;E</td>
</tr>
<tr>
<td>1,3</td>
<td>5</td>
<td>(\land)</td>
<td>3,4 (\neg E)</td>
</tr>
<tr>
<td>6</td>
<td>6</td>
<td>(\neg B)</td>
<td>Assumption ((\lor E))</td>
</tr>
<tr>
<td>1</td>
<td>7</td>
<td>(B)</td>
<td>1 &amp;E</td>
</tr>
<tr>
<td>1,6</td>
<td>8</td>
<td>(\land)</td>
<td>6,7 (\neg E)</td>
</tr>
<tr>
<td>1,2</td>
<td>9</td>
<td>(\land)</td>
<td>2,3,5,6,8 (\lor E)</td>
</tr>
<tr>
<td>1</td>
<td>10</td>
<td>(\neg(\neg A \lor \neg B))</td>
<td>2,9 (\neg I)</td>
</tr>
</tbody>
</table>
Proof of Example #6

Problem is: \( A \lor B \vdash \neg (\neg A \land \neg B) \)

1. \( A \lor B \)             \( \text{Premise} \)
2. \( \neg A \land \neg B \)         \( \text{Ass (}\neg I\text{)} \)
3. \( A \)                  \( \text{Ass (}\lor E\text{)} \)
2. \( \neg A \)              \( 2 \; \land E \)
2,3   \( \land \)                      \( 4,3 \; \neg E \)
6. \( B \)                  \( \text{Ass (}\lor E\text{)} \)
2. \( \neg B \)              \( 2 \; \land E \)
2,6   \( \land \)                      \( 7,6 \; \neg E \)
1,2   \( \land \)                      \( 1,3,5,6,8 \; \lor E \)
1. \( \neg (\neg A \land \neg B) \)   \( 2,9 \; \neg I \)
Proof of Example #7

Problem is: \( \neg(A \& B) \vdash \neg A \lor \neg B \)

1. \( \neg(A \& B) \) \hspace{1cm} \text{Premise}
2. \( \neg(\neg A \lor \neg B) \) \hspace{1cm} \text{Assumption (\neg I)}
3. \( \neg A \) \hspace{1cm} \text{Assumption (\neg I)}
3. \( \neg A \lor \neg B \) \hspace{1cm} 3 \lor I
2,3 \hspace{1cm} 5 \land \hspace{1cm} 2,4 \land E
2 \hspace{1cm} 6 \neg
8 \hspace{1cm} \neg B \) \hspace{1cm} 8 \neg I
8 \hspace{1cm} \neg A \lor \neg B \) \hspace{1cm} 8 \lor I
2,8 \hspace{1cm} 10 \land \hspace{1cm} 2,9 \land E
2 \hspace{1cm} 11 \neg
2 \hspace{1cm} 12 B \hspace{1cm} 11 \land I
2 \hspace{1cm} 7,12 \& I
1,2 \hspace{1cm} 14 \land \hspace{1cm} 1,13 \land E
1 \hspace{1cm} 15 \neg
1 \hspace{1cm} 2,14 \neg I
1 \hspace{1cm} 15 \land I
Proof of Example #8

Problem is: \( \sim C \vee (A \rightarrow B) \vdash (C \& A) \rightarrow B \)

1. (1) \( \sim C \vee (A \rightarrow B) \) Premise
2. (2) \( C \& A \) Assumption (\( \rightarrow I \))
3. (3) \( \sim B \) Assumption (\( \sim I \))
4. (4) \( \sim C \) Assumption (\( \sim E \))
2. (5) \( C \) \( \& E \)
2,4 (6) \( \Lambda \) \( 4,5 \sim E \)
7 (7) \( A \rightarrow B \) Assumption (\( \vee E \))
2 (8) \( A \) \( \& E \)
2,7 (9) \( B \) \( 7,8 \rightarrow E \)
2,3,7 (10) \( \Lambda \) \( 3,9 \sim E \)
1,2,3 (11) \( \Lambda \) \( 1,4,6,7,10 \vee E \)
1,2 (12) \( \sim \sim B \) \( 3,11 \sim I \)
1,2 (13) \( B \) \( 12 \ DN \)
1 (14) \( (C \& A) \rightarrow B \) \( 2,13 \rightarrow I \)
Proof of Example #9

Problem is: ⊢ (A → B) ∨ (B → A)

1
2
3
4
2
2
1,2
1,2
1
1
1

(1) \sim((A\rightarrow B) \lor (B\rightarrow A))
(2) B
(3) \sim A
(4) A
(5) A\rightarrow B
(6) (A\rightarrow B) \lor (B\rightarrow A)
(7) \sim A
(8) \sim A
(9) A
(10) B\rightarrow A
(11) (A\rightarrow B) \lor (B\rightarrow A)
(12) \sim A

Assumption (\sim I)
Assumption (\rightarrow I)
Assumption (\sim I)
Assumption (\rightarrow I)
4,2 \rightarrow I
5 \lor I
1,6 \sim E
3,7 \sim I
8 \lor I
2,9 \rightarrow I
10 \lor I
1,11 \sim E
1,12 \sim I
13 \lor I

DN
DN

Proof of Example #10

Problem is: \( \neg (A \lor B) \vdash \neg A \land \neg B \)

1. \( \neg (A \lor B) \)  
   \( \text{Premise} \)
2. \( A \)  
   \( \text{Ass (}\neg I\text{)} \)
2. \( A \lor B \)  
   \( 2 \lor I \)
1,2  \( \Delta \)  
   \( 1,3 \neg E \)
1  \( \neg A \)  
   \( 2,4 \neg I \)
6  \( B \)  
   \( \text{Ass (}\neg I\text{)} \)
6  \( A \lor B \)  
   \( 6 \lor I \)
1,6  \( \Delta \)  
   \( 1,7 \neg E \)
1  \( \neg B \)  
   \( 6,8 \neg I \)
1  \( \neg A \land \neg B \)  
   \( 5,9 \& I \)