Announcements & Such

- *Graham Nash*
- Administrative Stuff
  - **HW #3 solutions have been posted.**
  - **The Take-Home Mid-Term has been posted.**
  - **A Sample In-Class Mid-Term has been posted.**
  - ☞ **The Actual In-Class Mid-term is on Thursday.**
    - ∗ Bring blue-books and something to write with.
    - ∗ The exam is closed-book, closed-notes.
    - ∗ It shouldn't take you the full 75-minutes.
- Today: Chapter 4 — Natural Deduction Proofs for LSL
  - Continuing on with the LSL-*natural deduction rules* for $\vdash$.
  - **`MacLogic`** — a useful computer program for natural deduction.
    - ∗ See `http://fitelson.org/maclogic.htm`.
  - ☞ **Natural deductions are the most challenging topic of the course.**

How to Deduce a Conditional: I

- To deduce a conditional, we *assume* its antecedent and try to deduce its consequent from this assumption. If we are able to deduce the consequent from our assumption of the antecedent, then we *discharge* our assumption, and infer the conditional.
- To implement the $\rightarrow$I rule, we will first need a refined Rule of Assumptions that will allow us to assume arbitrary formulas "for the sake of argument", later to be discharged after making desired deductions. Here's the refined rule of Assumptions:
- **Rule of Assumptions** (final version): At any line j in a proof, any formula p may be entered and labeled as an assumption (or premise, where appropriate). The number j should then be written on the left. Schematically:

j	(j)	p	Assumption (or: Premise)

How to Deduce a Conditional: II — The →I Rule

- Now, we need a formal Introduction Rule for the →, which captures the intuitive idea sketched above (*i.e.*, assuming the antecedent, *etc.*):
- **Rule of →-Introduction**: For any formulae p and q, if q has been inferred at a line k in a proof and p is an assumption or premise occurring at line j, then at line m we may infer $\ulcorner p \to q \urcorner$, labeling the line 'j, k →I' and writing on the left the same assumption numbers which appear on the left of line k, except that we *delete* j if it is one of these numbers. Note: we may have j < k, j > k, *or* j = k (*why*?). Schematically:

j	(j)	p	Assumption (or: Premise)
	$\vdots$		
$a_1, \ldots, a_n$	(k)	q	
	$\vdots$		
$\{a_1, \ldots, a_n\}/j$	(m)	$p \to q$	j, k →I

Examples Involving &E, &I, →E, and →I

- Can you deduce the following, using &E, &I, →E, and →I?

(a) $A \to B$

$A \to C$

$\therefore A \to (B \,\&\, C)$

(b) $(A \,\&\, B) \to C$

$\therefore A \to (B \to C)$

(c) $B \,\&\, C$

$\therefore (A \to B) \,\&\, (A \to C)$

(d) $A \to B$

$\therefore (A \,\&\, C) \to (B \,\&\, C)$

(e) $A \,\&\, (B \,\&\, C)$

$\therefore A \to (B \to C)$

(f) $A \to B$

$\therefore A \to (C \to B)$

Important Tips For Using the →I Rule

- Use →I only when you wish to *derive* a conditional ⌜$p \to q$⌝.
- To derive ⌜$p \to q$⌝ using →I, assume the antecedent p and try to prove the consequent q. Always assume the *whole* of p, not just a part of it (like one of the conjuncts of a conjunction).
- When a conditional ⌜$p \to q$⌝ is derived by →I, the antecedent p must always be a formula which you have assumed at a previous line: it cannot be a formula that you have derived from other things. This is because it must be *discharged*.
- When you apply →I, remember to *discharge* the assumption by dropping the assumption number on the left.
- Check that the last line of your proof does not depend on any extra assumptions you have made besides your premises.

Proofs by Contradiction and the Rules for $\sim$

- If assuming p leads us to a contradiction, then we may infer $\ulcorner\sim p\urcorner$. [Note: This was implicit in our "short" truth-table method.]
- This style of proof is called *proof by contradiction* (or *reductio ad absurdum*). It is a very powerful technique that we'll see often.
- In our natural deduction system, the introduction and elimination rules for negation ($\sim$I and $\sim$E) allow us to perform *reductio*s.
- We use the symbol '⅄' to indicate that a contradiction has been deduced (*i.e.*, that p and $\ulcorner\sim p\urcorner$ have been deduced, for some p). We call '⅄' the *absurdity symbol* (an *atom*, added to the lexicon of LSL).
- With these preliminaries out of the way, we're ready to see what the negation rules look like, and how they work. . .

The Elimination Rule for ∼

Rule of ∼-Elimination: For any formula q, if ⌜∼q⌝ has been inferred at a line j in a proof and q at line k (j < k or j > k) then we may infer '⅄' at line m, labeling the line 'j, k ∼E' and writing on its left the numbers on the left at j and on the left at k. Schematically (with j < k):

$$
\begin{array}{rcll}
a_1, \ldots, a_n & (j) & \sim q & \\
 & \vdots & & \\
b_1, \ldots, b_u & (k) & q & \\
 & \vdots & & \\
a_1, \ldots, a_n, b_1, \ldots, b_u & (m) & ⅄ & \text{j, k} \sim\text{E}
\end{array}
$$

- Note: we have *added* the symbol '⅄' to the language of LSL. It is treated as if it were an *atomic sentence* of LSL. We can now use it in compound sentences (*e.g.*, '$A \to$ ⅄', '∼∼⅄', *etc*.).

The Introduction Rule for ∼

Rule of ∼-Introduction: If '⅄' has been inferred at line k in a proof and $\{a_1, \ldots, a_n\}$ are the assumption and premise numbers '⅄' depends upon, then if p is an assumption (or premise) at line j, ⌜$\sim p$⌝ may be inferred at line m, labeling the line 'j, k ∼I' and writing on its left the numbers in the set $\{a_1, \ldots, a_n\}/j$.

j	(j)	p	Assumption
	⋮		
$a_1, \ldots, a_n$	(k)	⅄	
	⋮		
$\{a_1, \ldots, a_n\}/j$	(m)	$\sim p$	j, k ∼I

- ∼I is used (typically *with* ∼E) to deduce ⌜$\sim p$⌝ *via reductio ad absurdum*, by (*i*) *assuming p*, (*ii*) deducing '⅄', and (*iii*) *discharging* the assumption.

Using The Rules ∼I and ∼E: Example #2

- Let's construct a proof of: $\sim(A \,\&\, B) \vdash (A \to \sim B)$.

1	(1)	$\sim(A \,\&\, B)$	Premise
2	(2)	A	Assumption
3	(3)	B	Assumption
2, 3	(4)	$A \,\&\, B$	2, 3 &I
1, 2, 3	(5)	$\curlywedge$	1, 4 ∼E
1, 2	(6)	$\sim B$	3, 5 ∼I
1	(7)	$A \to \sim B$	2, 6 →I ✦

The Rule of Double Negation (DN)

- Negation is an odd connective in our system. It not only has an introduction rule and an elimination rule, but it also has an additional rule called the *double negation* (DN) rule.
- The DN rule says that we may infer p from $\ulcorner{\sim}{\sim}p\urcorner$. Without this DN rule, we would not be able to prove certain valid LSL argument forms — *e.g.*, ${\sim}(A \,\&\, {\sim}B) \;\therefore\; (A \rightarrow B)$.

Rule of Double Negation: For any formula p, if $\ulcorner{\sim}{\sim}p\urcorner$ has been inferred at a line j in a proof, then at line k we may infer p, labeling the line 'j ' and writing on its left the numbers to the left of j.

$a_1,\ldots,a_n$	(j)	${\sim}{\sim}p$	
$a_1,\ldots,a_n$	(k)	p	j DN

An Example which *Requires* DN: I

- Consider the valid LSL form $\sim(A \,\&\, \sim B) \;\therefore\; (A \rightarrow B)$. If we try to prove this without using DN, we'll quickly get "stuck".
- We would begin by (*i*) writing down '$\sim(A \,\&\, \sim B)$' as our only Premise, then (*ii*) assuming 'A' and trying to deduce 'B'.
- But, since 'B' has no main connective, it's not clear how in the world we could possibly prove it. Without a main connective to introduce using an -I rule, we have no way to derive 'B'.
- But, '$\sim\sim B$' *does* have a main connective ('$\sim$'). So, we could use $\sim$I to prove '$\sim\sim B$', and then use DN to infer 'B'.
- In fact, this is the *only* strategy that will work!
- Let's prove $\sim(A \,\&\, \sim B) \vdash (A \rightarrow B)$.

An Example which *Requires* DN: II

1	(1)	$\sim(A \mathbin{\&} \sim B)$	Premise
2	(2)	A	Assumption
3	(3)	$\sim B$	Assumption
2, 3	(4)	$A \mathbin{\&} \sim B$	2, 3 &I
1, 2, 3	(5)	$\curlywedge$	1, 4 ~E
1, 2	(6)	$\sim\sim B$	3, 5 ~I
1, 2	(7)	B	6 DN
1	(8)	$A \to B$	2, 7 →I ✦

Another Example Requiring DN: Using MacLogic

- Here is a (MacLogic generated) proof of: $B, {\sim}B \vdash A$.

1	(1)	B	Premise
2	(2)	~B	Premise
3	(3)	~A	Assumption
1,2	(4)	$\curlywedge$	2,1 ~E
1,2	(5)	~~A	3,4 ~I
1,2	(6)	A	5 DN

Cautionary Remarks about *Reductio* Proofs

- Once you have deduced a contradiction (⅄) in the course of a proof, you can subsequently deduce *any* formula p *via* ~I and DN.
- But, such a deduction may depend on various assumptions, which means *they won't be proofs from the premises alone*. From last time:

1	(1)	$\sim(A \,\&\, B)$	Premise
2	(2)	A	Assumption
3	(3)	B	Assumption
2, 3	(4)	$A \,\&\, B$	2, 3 &I
1, 2, 3	(5)	⅄	1, 4 ~E
1, 2	(6)	$\sim B$	3, 5 ~I
1	(7)	$A \rightarrow \sim B$	2, 6 →I ✦

- You might be tempted to think that you could prove $A \to \sim B$ *via* $\sim$I and DN after step (5). You *can* deduce it in this way, *but* you get:

1	(1)	$\sim(A \,\&\, B)$	Premise
2	(2)	A	Assumption
3	(3)	B	Assumption
2, 3	(4)	$A \,\&\, B$	2, 3 &I
1, 2, 3	(5)	$\curlywedge$	1, 4 $\sim$E
6	(6)	$\sim(A \to \sim B)$	Assumption
1, 2, 3	(7)	$\sim\sim(A \to \sim B)$	6, 5 $\sim$I
1, 2, 3	(8)	$A \to \sim B$	7 DN

- *This does not help*. We need to prove $A \to \sim B$ from (1) *alone*, *not* from (1), (2), and (3). [Note: (1)–(3) is an *inconsistent* set!]
- Lesson: A strategy for proving the conclusion *from the premises alone* requires *discharging* all assumptions that are not premises.

Important Tips For Using the Negation Rules

- If you are trying to derive a formula with '$\sim$' as its main connective, use $\sim$I to obtain it. I.e., assume the formula within the scope of the '$\sim$' and try to derive $\curlywedge$ using $\sim$E.
- When you apply $\sim$I, the formula which you infer must be the negation of a premise or an assumption. It cannot be the negation of a formula which has been deduced.
- If one of your premises or assumptions has '$\sim$' as its main connective, it is likely that its role in the proof will be to be one of a pair of contradictory formulae in an application of $\sim$E. You should therefore consider trying to derive the formula within the scope of the '$\sim$' to get the other member of the contradictory pair.
- If you are trying to deduce a sentence-letter and there is no obvious way to do it, consider trying to derive its double-negation and then use DN. Last Resort!
- At this point, *you should only assume a formula p if you are trying to deduce its negation or trying to deduce a conditional with p as antecedent*. Only make an assumption when you've figured how you're going to use $\sim$I or $\rightarrow$I to *discharge* it.

General Strategy — Working in Both Directions

- Begin by writing the premises (if any) at the top of your scratch paper area, using the Rule of Assumptions.
- Then, write the conclusion (the *main* goal formula) you're trying to derive at the bottom of your scratch area.
- Next, determine what the main connective (if any) of your conclusion is, then apply the introduction rule for that connective.
- This will yield *sub-goal* formula(s). Write the sub-goal formula(s) directly above your conclusion. Then try to figure-out how to prove the sub-goal formula(s) from your premises.
- This will yield sub-sub-goal formula(s). And so on …
- Repeat this process until you have worked your way all the way back up to your premises/assumptions (if a formula is resisting proof, you might try to prove its *double-negation* using ∼I with ∼E).

Example Proof of a *Theorem*

- Using only the rules we have learned so far, we should be able to prove the following *theorem*: $\vdash \sim(A \,\&\, {\sim}A)$. Let's do this one by hand first.
- Here's a simple proof, generated using MacLogic (I'll show how):

Problem is: $\vdash$ ~(A&~A)

1	(1)	A&~A	Assumption (!)
1	(2)	~A	1 &E
1	(3)	A	1 &E
1	(4)	$\wedge$	2,3 ~E
	(5)	~(A&~A)	1,4 ~I

- This proof makes use of *no premises*, and its final line has *no numbers to its left* — indicating that we have succeeded in proving '$\sim(A \,\&\, {\sim}A)$' from *nothing at all*. It's a *theorem* (*i.e.*, a sequent with no premises)!

The Introduction Rule for ∨ (∨I)

Rule of ∨-Introduction: For any formula p, if p has been inferred at line j, then, for any formula q, *either* $\ulcorner p \vee q \urcorner$ *or* $\ulcorner q \vee p \urcorner$ may be inferred at line k, labeling the line 'j ∨I' and writing on its left the same premise and assumption numbers as appear on the left of j.

$a_1,\ldots,a_n$	(j)	p			$a_1,\ldots,a_n$	(j)	q	
	$\vdots$			OR		$\vdots$		
$a_1,\ldots,a_n$	(k)	$p \vee q$	j ∨I		$a_1,\ldots,a_n$	(k)	$p \vee q$	j ∨I

- The ∨I rule is very simple an intuitive. Basically, it says that you may infer a disjunction from *either* of its disjuncts.
- The *elimination* rule (∨E) for ∨, on the other hand, is considerably more complex to state and apply. It's the hardest of our rules.

The Elimination Rule for $\vee$ ($\vee$E)

- First, the idea *behind* the $\vee$-elimination rule.
- The following argument form is valid (easily verified *via* truth-table):

$$
\begin{array}{l}
p \vee q \\
p \to r \\
q \to r \\
\therefore r
\end{array}
$$

- This argument form is called the *constructive dilemma*. In essence, the $\vee$E rule reflects the constructive dilemma form of reasoning and implements it in our system of natural deduction rules.
- The $\vee$E rule is trickier than our other rules because it requires us to make *two* assumptions. This can make it rather complicated to keep track of all of our assumptions and premises during an $\vee$E proof.
- Now, the official definition of $\vee$E …

Rule of ∨-Elimination: If a disjunction ⌜$p \vee q$⌝ occurs at line g of a proof, p is assumed at line h, r is derived at line i, q is assumed at line j, and r is derived at line k, then at line m we may infer r, labeling the line 'g, h, i, j, k ∨E' and writing on its left every number on the left at line g, and at line i (except h), and at line k (except j).

$a_1, \ldots, a_n$	(g)	$p \vee q$	
	⋮		
h	(h)	p	Assumption
	⋮		
$b_1, \ldots, b_u$	(i)	r	
	⋮		
j	(j)	q	Assumption
	⋮		
$c_1, \ldots, c_w$	(k)	r	
	⋮		
$\mathscr{A}$	(m)	r	g, h, i, j, k ∨E

where $\mathscr{A}$ is the set: $\{a_1, \ldots, a_n\} \cup \{b_1, \ldots, b_u\}/h \cup \{c_1, \ldots, c_w\}/j$.

An Example Involving $\vee$E and DN

- Here's a proof of the sequent: $A \vee B, {\sim}B \vdash A$.

Problem is: $A \vee B, {\sim}B \vdash A$

1	(1)	$A \vee B$	Premise
2	(2)	${\sim}B$	Premise
3	(3)	${\sim}A$	Assumption (for ~I)
4	(4)	A	Assumption (for $\vee$E)
3,4	(5)	$\curlywedge$	3,4 ~E
6	(6)	B	Assumption (for $\vee$E)
2,6	(7)	$\curlywedge$	2,6 ~E
1,2,3	(8)	$\curlywedge$	1,4,5,6,7 $\vee$E
1,2	(9)	${\sim}{\sim}A$	3,8 ~I
1,2	(10)	A	9 DN

A Simple Example Involving $\vee$I and $\vee$E

- Here's a proof of the sequent: $A \vee B \vdash B \vee A$.

Problem is: $A \vee B \vdash B \vee A$

1	(1)	$A \vee B$	Premise
2	(2)	A	Assumption ($\vee$E)
2	(3)	$B \vee A$	2 $\vee$I
4	(4)	B	Assumption ($\vee$E)
4	(5)	$B \vee A$	4 $\vee$I
1	(6)	$B \vee A$	1,2,3,4,5 $\vee$E

Another Example Involving $\vee$I and Negation

- Here's a proof of the *theorem*: $\vdash A \vee \sim A$.

Problem is: $\vdash A \vee \sim A$

1	(1)	$\sim(A \vee \sim A)$	Assumption ($\sim$I)
2	(2)	A	Assumption ($\sim$I)
2	(3)	$A \vee \sim A$	2 $\vee$I
1,2	(4)	$\wedge$	1,3 $\sim$E
1	(5)	$\sim A$	2,4 $\sim$I
1	(6)	$A \vee \sim A$	5 $\vee$I
1	(7)	$\wedge$	1,6 $\sim$E
	(8)	$\sim\sim(A \vee \sim A)$	1,7 $\sim$I
	(9)	$A \vee \sim A$	8 DN