Announcements & Such

- *Townes Van Zandt*
- Administrative Stuff
  - **HW #3 resubmissions are due today.**
    - ☞ **When you turn in resubmissions, make sure that you staple them to your original homework submission.**
  - **The Take-Home Mid-Term has been posted.** [Discuss.]
  - **A Sample In-Class Mid-Term has been posted.** [Discuss.]
  - ☞ **The Actual In-Class Mid-term is next Thursday, 3/11.**
- Today: Chapter 4 — Natural Deduction Proofs for LSL
  - Continuing on with the LSL-*natural deduction rules* for $\vdash$.
  - **`MacLogic`** — a useful computer program for natural deduction.
    - ∗ See `http://fitelson.org/maclogic.htm`.
  - ☞ **Natural deductions are the most challenging topic of the course.**

The Rule of Assumptions (Preliminary Version)

- **Rule of Assumptions** (preliminary version): The premises of an argument-form are listed at the start of a proof in the order in which they are given, each labeled 'Premise' on the right and numbered with its own line number on the left. Schematically:

j	(j)	p	Premise

The Rule of &-Elimination (&E)

- **Rule of &-Elimination**: If a conjunction ⌜$p \,\&\, q$⌝ occurs at line j, then at any *later* line k one may infer either conjunct, labeling the line 'j &E' and writing on the left all the numbers which appear on the left of line j.

Schematically:

$a_1, \ldots, a_n$	(j)	$p \,\&\, q$			$a_1, \ldots, a_n$	(j)	$p \,\&\, q$	
	$\vdots$			OR		$\vdots$		
$a_1, \ldots, a_n$	(k)	p	j &E		$a_1, \ldots, a_n$	(k)	q	j &E

The Rule of &-Introduction (&I)

- **Rule of &-Introduction**: For any formulae p and q, if p occurs at line j and q occurs at line k then the formula $\ulcorner p \,\&\, q \urcorner$ may be inferred at line m, labeling the line 'j, k &I' and writing on the left all numbers which appear on the left of line j *and* all which appear on the left of line k. [Note: we may have j < k, j > k, *or* j = k. *Why*?]

$$
\begin{array}{rcll}
a_1,\ldots,a_n & (j) & p & \\
 & \vdots & & \\
b_1,\ldots,b_u & (k) & q & \\
 & \vdots & & \\
a_1,\ldots,a_n, b_1,\ldots,b_u & (m) & p \,\&\, q & \text{j, k \&I}
\end{array}
$$

The Rule of →-Elimination (→E)

- **Rule of →-Elimination**: For any formulae p and q, if ⌜$p \to q$⌝ occurs at a line j and p occurs at a line k, then q may be inferred at line m, labeling the line 'j, k →E' and writing on the left all numbers which appear on the left of line j *and* all numbers which appear on the left of line k.
  [Note: We may have either j < k or j > k.]

$a_1, \ldots, a_n$	(j)	$p \to q$	
	$\vdots$		
$b_1, \ldots, b_u$	(k)	p	
	$\vdots$		
$a_1, \ldots, a_n, b_1, \ldots, b_u$	(m)	q	j, k →E

Deduction #3 Using the Rules &E, &I, and →E

- Let's do a deduction of:

$$A \to (B \to (C \to D))$$
$$C \,\&\, (A \,\&\, B)$$
$$\therefore D$$

1	(1)	$A \to (B \to (C \to D))$	Premise
2	(2)	$C \,\&\, (A \,\&\, B)$	Premise
2	(3)	$A \,\&\, B$	2 &E
2	(4)	A	3 &E
1, 2	(5)	$B \to (C \to D)$	1, 4 →E
2	(6)	B	3 &E
1, 2	(7)	$C \to D$	5, 6 →E
1	(8)	C	2 &E
1, 2	(9)	D	7, 8 →E ✦

Note on -E Rules — Avoiding a Common Error

- As with &E, $\rightarrow$E can *only* be applied to the *main* $\rightarrow$ of a conditional — *not* to any *other* $\rightarrow$'s which may be in a formula.
- So, the step from (3) to (4) in the following is *incorrect*.

1	(1)	$A \rightarrow (B \rightarrow (C \rightarrow D))$	Premise
2	(2)	$C \,\&\, (A \,\&\, B)$	Premise
2	(3)	C	2 &E
1, 2	(4)	D	1, 3 $\rightarrow$E (NO!)

- *The elimination rule for a connective c can* ***only*** *be applied to a line if that line has an occurrence of c as its* ***main*** *connective, and the rule* ***must*** *be applied to* ***that*** *occurrence of c.*

How to Deduce a Conditional: I

- To deduce a conditional, we *assume* its antecedent and try to deduce its consequent from this assumption. If we are able to deduce the consequent from our assumption of the antecedent, then we *discharge* our assumption, and infer the conditional.
- To implement the $\to$I rule, we will first need a refined Rule of Assumptions that will allow us to assume arbitrary formulas "for the sake of argument", later to be discharged after making desired deductions. Here's the refined rule of Assumptions:
- **Rule of Assumptions** (final version): At any line j in a proof, any formula p may be entered and labeled as an assumption (or premise, where appropriate). The number j should then be written on the left. Schematically:

j	(j)	p	Assumption (or: Premise)

How to Deduce a Conditional: II — The →I Rule

- Now, we need a formal Introduction Rule for the →, which captures the intuitive idea sketched above (*i.e.*, assuming the antecedent, *etc.*):
- **Rule of →-Introduction**: For any formulae p and q, if q has been inferred at a line k in a proof and p is an assumption or premise occurring at line j, then at line m we may infer $\ulcorner p \to q \urcorner$, labeling the line 'j, k →I' and writing on the left the same assumption numbers which appear on the left of line k, except that we *delete* j if it is one of these numbers. Note: we may have j < k, j > k, *or* j = k (*why*?). Schematically:

j	(j)	p	Assumption (or: Premise)
	⋮		
$a_1, \ldots, a_n$	(k)	q	
	⋮		
$\{a_1, \ldots, a_n\}/j$	(m)	$p \to q$	j, k →I

Using The →I Rule: An Example

- Let's do a deduction of: $A \to (B \to C)$
  $\therefore (A \to B) \to (A \to C)$

1	(1)	$A \to (B \to C)$	Premise
2	(2)	$A \to B$	Assumption
3	(3)	A	Assumption
2, 3	(4)	B	2, 3 →E
1, 3	(5)	$B \to C$	1, 3 →E
1, 2, 3	(6)	C	4, 5 →E
1, 2	(7)	$A \to C$	3, 6 →I
1	(8)	$(A \to B) \to (A \to C)$	2, 7 →I ✦

Examples Involving &E, &I, →E, and →I

- Can you deduce the following, using &E, &I, →E, and →I?

(a)
$A \to B$
$A \to C$
$\therefore A \to (B \,\&\, C)$

(b)
$(A \,\&\, B) \to C$
$\therefore A \to (B \to C)$

(c)
$B \,\&\, C$
$\therefore (A \to B) \,\&\, (A \to C)$

(d)
$A \to B$
$\therefore (A \,\&\, C) \to (B \,\&\, C)$

(e)
$A \,\&\, (B \,\&\, C)$
$\therefore A \to (B \to C)$

(f)
$A \to B$
$\therefore A \to (C \to B)$

One[a] Solution to (c) (*not* solved in the text)

	1	(1)	$B \& C$	Premise
	2	(2)	A	Assumption
	1	(3)	B	1 &E
(c)	1	(4)	C	1 &E
	1	(5)	$A \rightarrow B$	2, 3 →I
	1	(6)	$A \rightarrow C$	2, 4 →I
	1	(7)	$(A \rightarrow B) \& (A \rightarrow C)$	5, 6 &I ✦

[a] There are many, many correct deductions of any valid argument.

One Solution to (d) (*not* in the text)

	1	(1)	$A \to B$	Premise
	2	(2)	$A \,\&\, C$	Assumption
	2	(3)	A	2 &E
(d)	2	(4)	C	2 &E
	1, 2	(5)	B	1, 3 →E
	1, 2	(6)	$B \,\&\, C$	5, 4 &I
	1	(7)	$(A \,\&\, C) \to (B \,\&\, C)$	2, 6 →I ✦

Important Tips For Using the →I Rule

- Use →I only when you wish to *derive* a conditional ⌜$p \to q$⌝.
- To derive ⌜$p \to q$⌝ using →I, assume the antecedent p and try to prove the consequent q. Always assume the *whole* of p, not just a part of it (like one of the conjuncts of a conjunction).
- When a conditional ⌜$p \to q$⌝ is derived by →I, the antecedent p must always be a formula which you have assumed at a previous line: it cannot be a formula that you have derived from other things. This is because it must be *discharged*.
- When you apply →I, remember to *discharge* the assumption by dropping the assumption number on the left.
- Check that the last line of your proof does not depend on any extra assumptions you have made besides your premises.

Proofs by Contradiction and the Rules for $\sim$

- If assuming p leads us to a contradiction, then we may infer $\ulcorner \sim p \urcorner$. [Note: This was implicit in our "short" truth-table method.]
- This style of proof is called *proof by contradiction* (or *reductio ad absurdum*). It is a very powerful technique that we'll see often.
- In our natural deduction system, the introduction and elimination rules for negation ($\sim$I and $\sim$E) allow us to perform *reductio*s.
- We use the symbol '$\curlywedge$' to indicate that a contradiction has been deduced (*i.e.*, that p and $\ulcorner \sim p \urcorner$ have been deduced, for some p). We call '$\curlywedge$' the *absurdity symbol* (an *atom*, added to the lexicon of LSL).
- With these preliminaries out of the way, we're ready to see what the negation rules look like, and how they work…

The Elimination Rule for ∼

Rule of ∼-Elimination: For any formula q, if ⌜∼q⌝ has been inferred at a line j in a proof and q at line k (j < k or j > k) then we may infer '⋏' at line m, labeling the line 'j, k ∼E' and writing on its left the numbers on the left at j and on the left at k. Schematically (with j < k):

$a_1,\ldots,a_n$	(j)	$\sim q$	
	⋮		
$b_1,\ldots,b_u$	(k)	q	
	⋮		
$a_1,\ldots,a_n, b_1,\ldots,b_u$	(m)	⋏	j, k ∼E

- Note: we have *added* the symbol '⋏' to the language of LSL. It is treated as if it were an *atomic sentence* of LSL. We can now use it in compound sentences (*e.g.*, '$A \rightarrow$ ⋏', '∼∼⋏', *etc.*).

The Introduction Rule for $\sim$

Rule of $\sim$-Introduction: If '$\curlywedge$' has been inferred at line k in a proof and $\{a_1, \ldots, a_n\}$ are the assumption and premise numbers '$\curlywedge$' depends upon, then if p is an assumption (or premise) at line j, $\ulcorner \sim p \urcorner$ may be inferred at line m, labeling the line 'j, k $\sim$I' and writing on its left the numbers in the set $\{a_1, \ldots, a_n\}/j$.

j	(j)	p	Assumption
	$\vdots$		
$a_1, \ldots, a_n$	(k)	$\curlywedge$	
	$\vdots$		
$\{a_1, \ldots, a_n\}/j$	(m)	$\sim p$	j, k $\sim$I

- $\sim$I is used (typically *with* $\sim$E) to deduce $\ulcorner \sim p \urcorner$ *via reductio ad absurdum*, by (*i*) *assuming* p, (*ii*) deducing '$\curlywedge$', and (*iii*) *discharging* the assumption.

Using The Rules ∼I and ∼E: Example #2

- Let's construct a proof of: $\sim(A \,\&\, B) \vdash (A \rightarrow \sim B)$.

1	(1)	$\sim(A \,\&\, B)$	Premise
2	(2)	A	Assumption
3	(3)	B	Assumption
2, 3	(4)	$A \,\&\, B$	2, 3 &I
1, 2, 3	(5)	$\curlywedge$	1, 4 ∼E
1, 2	(6)	$\sim B$	3, 5 ∼I
1	(7)	$A \rightarrow \sim B$	2, 6 →I ✦

The Rule of Double Negation (DN)

- Negation is an odd connective in our system. It not only has an introduction rule and an elimination rule, but it also has an additional rule called the *double negation* (DN) rule.
- The DN rule says that we may infer p from $\ulcorner{\sim}{\sim}p\urcorner$. Without this DN rule, we would not be able to prove certain valid LSL argument forms — *e.g.*, ${\sim}(A \mathbin{\&} {\sim}B) \therefore (A \rightarrow B)$.

Rule of Double Negation: For any formula p, if $\ulcorner{\sim}{\sim}p\urcorner$ has been inferred at a line j in a proof, then at line k we may infer p, labeling the line 'j ' and writing on its left the numbers to the left of j.

$a_1, \ldots, a_n$	(j)	${\sim}{\sim}p$	
$a_1, \ldots, a_n$	(k)	p	j DN

An Example which *Requires* DN: I

- Consider the valid LSL form $\sim(A \,\&\, \sim B) \;\therefore\; (A \to B)$. If we try to prove this without using DN, we'll quickly get "stuck".
- We would begin by (*i*) writing down '$\sim(A \,\&\, \sim B)$' as our only Premise, then (*ii*) assuming 'A' and trying to deduce 'B'.
- But, since 'B' has no main connective, it's not clear how in the world we could possibly prove it. Without a main connective to introduce using an -I rule, we have no way to derive 'B'.
- But, '$\sim\sim B$' *does* have a main connective ('$\sim$'). So, we could use $\sim$I to prove '$\sim\sim B$', and then use DN to infer 'B'.
- In fact, this is the *only* strategy that will work!
- Let's prove $\sim(A \,\&\, \sim B) \vdash (A \to B)$.

An Example which *Requires* DN: II

1	(1)	$\sim(A \,\&\, \sim B)$	Premise
2	(2)	A	Assumption
3	(3)	$\sim B$	Assumption
2, 3	(4)	$A \,\&\, \sim B$	2, 3 &I
1, 2, 3	(5)	$\curlywedge$	1, 4 ~E
1, 2	(6)	$\sim\sim B$	3, 5 ~I
1, 2	(7)	B	6 DN
1	(8)	$A \rightarrow B$	2, 7 →I ✦

Another Example Requiring DN: Using MacLogic

- Here is a (MacLogic generated) proof of: $B, \sim B \vdash A$.

1	(1)	B	Premise
2	(2)	~B	Premise
3	(3)	~A	Assumption
1,2	(4)	$\wedge$	2,1 ~E
1,2	(5)	~~A	3,4 ~I
1,2	(6)	A	5 DN

Cautionary Remarks about *Reductio* Proofs

- Once you have deduced a contradiction (⅄) in the course of a proof, you can subsequently deduce *any* formula p *via* ~I and DN.
- But, such a deduction may depend on various assumptions, which means *they won't be proofs from the premises alone*. From last time:

1	(1)	$\sim(A \,\&\, B)$	Premise
2	(2)	A	Assumption
3	(3)	B	Assumption
2, 3	(4)	$A \,\&\, B$	2, 3 &I
1, 2, 3	(5)	⅄	1, 4 ~E
1, 2	(6)	$\sim B$	3, 5 ~I
1	(7)	$A \to \sim B$	2, 6 →I ✦

- You might be tempted to think that you could prove $A \to \sim B$ *via* $\sim$I and DN after step (5). You *can* deduce it in this way, *but* you get:

1	(1)	$\sim(A \& B)$	Premise
2	(2)	A	Assumption
3	(3)	B	Assumption
2, 3	(4)	$A \& B$	2, 3 &I
1, 2, 3	(5)	$\curlywedge$	1, 4 $\sim$E
6	(6)	$\sim(A \to \sim B)$	Assumption
1, 2, 3	(7)	$\sim\sim(A \to \sim B)$	6, 5 $\sim$I
1, 2, 3	(8)	$A \to \sim B$	7 DN

- *This does not help*. We need to prove $A \to \sim B$ from (1) *alone*, *not* from (1), (2), and (3). [Note: (1)–(3) is an *inconsistent* set!]
- Lesson: A strategy for proving the conclusion *from the premises alone* requires *discharging* all assumptions that are not premises.