Announcements & Such

- *Air*: *La Femme D'Argent*
- Administrative Stuff
  - ☞ **It is *business-as-usual* in 12A this week — *despite* the strike.**
  - HW #3 will be returned Today — Resubs due Thursday.
    - ☞ **When you turn in resubmissions, make sure that you staple them to your original homework submission.**
  - **The Take-Home Mid-Term will be posted on Thursday.**
  - **A Sample In-Class Mid-Term will be posted on Thursday.**
  - ☞ **The Actual In-Class Mid-term is next Thursday, 3/11.**
- Today: Chapter 4 — Natural Deduction Proofs for LSL
  - Validity ($\vDash$) *vs* Proof ($\vdash$), and the LSL *rules* for $\vdash$.
  - **`MacLogic`** — a useful computer program for natural deduction.
    - ∗ See `http://fitelson.org/maclogic.htm`.
  - ☞ **Natural deductions are the most challenging topic of the course.**

Expressive Completeness: Rewind, and More Extra-Credit

- **Q.** How can we define $\leftrightarrow$ in terms of $|$? **A.** If you naïvely apply the schemes I described last time, then you get a *187 symbol monster*:

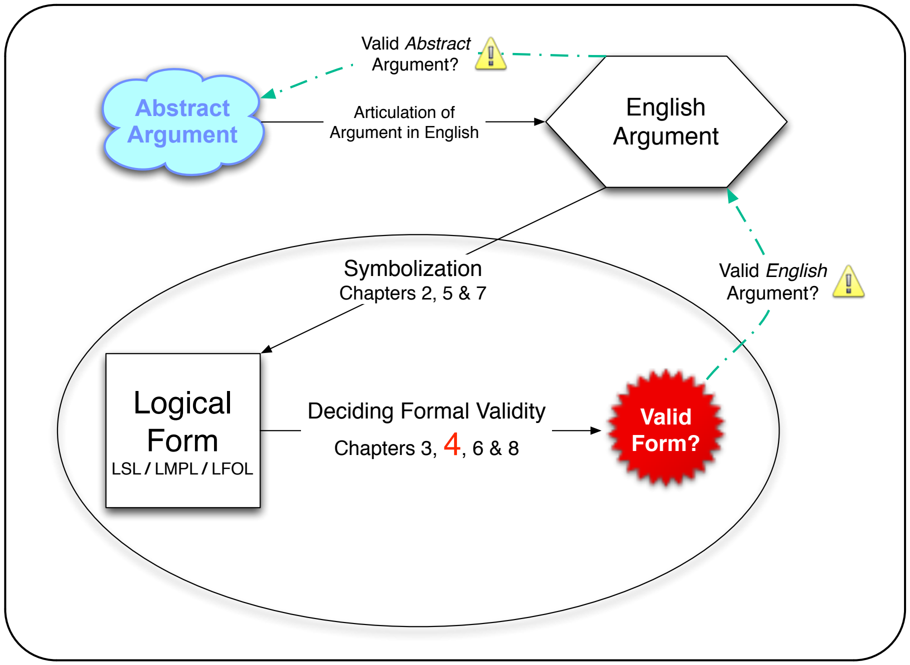
  $\ulcorner p \leftrightarrow q \urcorner \mapsto A|A$, where A is given by the following *93 symbol* expression:

  $$(((p|(q|q))|(p|(q|q)))|((p|(q|q))|(p|(q|q))))|(((q|(p|p))|(q|(p|p)))|((q|(p|p))|(q|(p|p))))$$

- There are *simpler* definitions of $\leftrightarrow$ using $|$. *E.g.*, this *43 symbol* answer:

  $$\ulcorner p \leftrightarrow q \urcorner \mapsto ((p|(q|q))|(q|(p|p)))|((p|(q|q))|(q|(p|p)))$$

- I offered E.C. for a shorter solution. Some students came up with a 19-symbol solution (counting parens), which is the *shortest possible*.
- **More E.C.** Find the *shortest possible* definitions of (1) $\ulcorner p \to q \urcorner$, (2) $\ulcorner p \vee q \urcorner$, and (3) $\ulcorner \sim p \,\&\, \sim q \urcorner$ in terms of p, q, and the NAND operator $|$.
- If you submit EC, please *prove* the correctness of your solution, using a truth-table method. You may submit these E.C. solutions to your GSI.



Chapter 4 Introduction: Truth *vs* Proof ($\vDash$ *vs* $\vdash$)

- Recall: $p \vDash q$ iff it is impossible for p to be true while q is false.
- We have methods (truth-tables) for establishing $\vDash$ and $\nvDash$ claims. These methods are especially good for $\nvDash$ claims, but they get very complex for $\vDash$ claims. Is there another more "natural" way to prove $\vDash$'s? Yes!
- In Chapter 4, we will learn a *natural deduction system* for LSL. This is a system of *rules of inference* that will allow us to prove all valid LSL arguments in a purely syntactical way (no appeal to semantics).
- The notation $p \vdash q$ means that *there exists a natural deduction proof of q from p* in our natural deduction system for sentential logic.
- $\ulcorner p \vdash q \urcorner$ is short for $\ulcorner p$ *deductively* entails $q \urcorner$.
- While $\vDash$ has to do with *truth*, $\vdash$ does *not*. $\vdash$ has only to do with what can be *deduced*, using a *fixed set* of formal, natural deduction rules.

- Happily, our system of natural deduction rules is *sound* and *complete*:
  - **Soundness**. If $p \vdash q$, then $p \vDash q$. [no proofs of *in*validities!]
  - **Completeness**. If $p \vDash q$, then $p \vdash q$. [proofs of *all* validities!]
- We will not prove the soundness and completeness of our system of natural deduction rules. I will say a few things about soundness as we go along, but completeness is much harder to establish (140A!).
- We'll have rules that permit the *elimination* or *introduction* of each of the connectives &, →, ∨, ~, ↔ within natural deductions. These rules will make sense, from the point of view of the semantics.
- A *proof* of q from p is a sequence of LSL formulas, beginning with p and ending with q, where each formula in the sequence is *deduced* from previous lines, *via* a correct application of one of the *rules*.
- Generally, we will be talking about deductions of formulas q from sets of premises $p_1, \ldots, p_n$. We call these $\ulcorner p_1, \ldots, p_n \vdash q \urcorner$'s *sequents*.

An Example of a Natural Deduction Involving & and →

- The following is a valid LSL argument form:

  $A \,\&\, B$
  $C \,\&\, D$
  $(A \,\&\, D) \to H$
  $\therefore H$

- Here's a (7-line) natural deduction proof of the sequent corresponding to this argument: $A \,\&\, B, C \,\&\, D, (A \,\&\, D) \to H \vdash H$.

1	(1)	$A \,\&\, B$	Premise
2	(2)	$C \,\&\, D$	Premise
3	(3)	$(A \,\&\, D) \to H$	Premise
1	(4)	A	1 &E
2	(5)	D	2 &E
1, 2	(6)	$A \,\&\, D$	4, 5 &I
1, 2, 3	(7)	H	3, 6 →E ✦

The Rule of Assumptions (Preliminary Version)

- **Rule of Assumptions** (preliminary version): The premises of an argument-form are listed at the start of a proof in the order in which they are given, each labeled 'Premise' on the right and numbered with its own line number on the left. Schematically:

j	(j)	p	Premise

- We can see that our example proof begins, as it should, with the three premises of the argument-form, written as follows:

1	(1)	$A \,\&\, B$	Premise
2	(2)	$C \,\&\, D$	Premise
3	(3)	$(A \,\&\, D) \to H$	Premise

The Rule of &-Elimination (&E)

- **Rule of &-Elimination**: If a conjunction $\ulcorner p \,\&\, q \urcorner$ occurs at line j, then at any *later* line k one may infer either conjunct, labeling the line 'j &E' and writing on the left all the numbers which appear on the left of line j.

  Schematically:

$a_1, \ldots, a_n$	(j)	$p \,\&\, q$			$a_1, \ldots, a_n$	(j)	$p \,\&\, q$	
	⋮			OR		⋮		
$a_1, \ldots, a_n$	(k)	p	j &E		$a_1, \ldots, a_n$	(k)	q	j &E

- We can see that our example deduction continues, in lines (4) and (5), with two correct applications of the &-Elimination Rule:

1	(4)	A	1 &E
2	(5)	D	2 &E

The Rule of &-Introduction (&I)

- **Rule of &-Introduction**: For any formulae p and q, if p occurs at line j and q occurs at line k then the formula ⌜$p \,\&\, q$⌝ may be inferred at line m, labeling the line 'j, k &I' and writing on the left all numbers which appear on the left of line j *and* all which appear on the left of line k. [Note: we may have j < k, j > k, *or* j = k. *Why*?]

$$
\begin{array}{lll}
a_1,\ldots,a_n & (j) & p \\
 & \vdots & \\
b_1,\ldots,b_u & (k) & q \\
 & \vdots & \\
a_1,\ldots,a_n,b_1,\ldots,b_u & (m) & p \,\&\, q \quad \text{j, k \&I}
\end{array}
$$

- We can see that our example deduction continues, in lines (6), with a correct application of the &-Introduction Rule:

1, 2 (6) $A \,\&\, D$ 4, 5 &I

The Rule of →-Elimination (→E)

- **Rule of →-Elimination**: For any formulae p and q, if ⌜$p \to q$⌝ occurs at a line j and p occurs at a line k, then q may be inferred at line m, labeling the line 'j, k →E' and writing on the left all numbers which appear on the left of line j *and* all numbers which appear on the left of line k. [Note: We may have either j < k or j > k.]

$$
\begin{array}{lll}
a_1,\ldots,a_n & (j) & p \to q \\
 & \vdots & \\
b_1,\ldots,b_u & (k) & p \\
 & \vdots & \\
a_1,\ldots,a_n,b_1,\ldots,b_u & (m) & q \quad \text{j, k} \to\text{E}
\end{array}
$$

- Our example deduction *concludes* (we indicate the end of a proof with a '✦'), in line (7), with a correct application of the →-Elimination Rule:

1, 2, 3 (7) H 3, 6 →E ✦

Deduction #2 Using the Rules &E and &I

- Consider the valid LSL argument form: $A \,\&\, (B \,\&\, C)$ $\therefore C \,\&\, (B \,\&\, A)$
- Let's do a deduction of this argument form:

1	(1)	$A \,\&\, (B \,\&\, C)$	Premise
1	(2)	A	1 &E
1	(3)	$B \,\&\, C$	1 &E
1	(4)	B	3 &E
1	(5)	C	3 &E
1	(6)	$B \,\&\, A$	4, 2 &I
1	(7)	$C \,\&\, (B \,\&\, A)$	5, 6 &I ✦

- NOTE: &E can *only* be applied to formulas whose *main* connective is '&', and &E *must* be *applied to that particular* connective.

Deduction #3 Using the Rules &E, &I, and →E

- Let's do a deduction of: $A \to (B \to (C \to D))$ $C \,\&\, (A \,\&\, B)$ $\therefore D$

1	(1)	$A \to (B \to (C \to D))$	Premise
2	(2)	$C \,\&\, (A \,\&\, B)$	Premise
2	(3)	$A \,\&\, B$	2 &E
2	(4)	A	3 &E
1, 2	(5)	$B \to (C \to D)$	1, 4 →E
2	(6)	B	3 &E
1, 2	(7)	$C \to D$	5, 6 →E
1	(8)	C	2 &E
1, 2	(9)	D	7, 8 →E ✦

Note on -E Rules — Avoiding a Common Error

- As with &E, →E can *only* be applied to the *main* → of a conditional — *not* to any *other* →'s which may be in a formula.
- So, the step from (3) to (4) in the following is *incorrect*.

1	(1)	$A \to (B \to (C \to D))$	Premise
2	(2)	$C \,\&\, (A \,\&\, B)$	Premise
2	(3)	C	2 &E
1, 2	(4)	D	1, 3 →E (NO!)

- *The elimination rule for a connective c can **only** be applied to a line if that line has an occurrence of c as its **main** connective, and the rule **must** be applied to **that** occurrence of c.*

How to Deduce a Conditional: I

- To deduce a conditional, we *assume* its antecedent and try to deduce its consequent from this assumption. If we are able to deduce the consequent from our assumption of the antecedent, then we *discharge* our assumption, and infer the conditional.
- To implement the →I rule, we will first need a refined Rule of Assumptions that will allow us to assume arbitrary formulas "for the sake of argument", later to be discharged after making desired deductions. Here's the refined rule of Assumptions:
- **Rule of Assumptions** (final version): At any line j in a proof, any formula p may be entered and labeled as an assumption (or premise, where appropriate). The number j should then be written on the left. Schematically:

j	(j)	p	Assumption (or: Premise)

How to Deduce a Conditional: II — The →I Rule

- Now, we need a formal Introduction Rule for the →, which captures the intuitive idea sketched above (*i.e.*, assuming the antecedent, *etc.*):
- **Rule of →-Introduction**: For any formulae p and q, if q has been inferred at a line k in a proof and p is an assumption or premise occurring at line j, then at line m we may infer ⌜$p \to q$⌝, labeling the line 'j, k →I' and writing on the left the same assumption numbers which appear on the left of line k, except that we *delete* j if it is one of these numbers. Note: we may have j < k, j > k, *or* j = k (*why*?). Schematically:

j	(j)	p	Assumption (or: Premise)
	⋮		
$a_1, \ldots, a_n$	(k)	q	
	⋮		
$\{a_1, \ldots, a_n\}/j$	(m)	$p \to q$	j, k →I

Using The →I Rule: An Example

- Let's do a deduction of: $A \to (B \to C)$ $\therefore (A \to B) \to (A \to C)$

1	(1)	$A \to (B \to C)$	Premise
2	(2)	$A \to B$	Assumption
3	(3)	A	Assumption
2, 3	(4)	B	2, 3 →E
1, 3	(5)	$B \to C$	1, 3 →E
1, 2, 3	(6)	C	4, 5 →E
1, 2	(7)	$A \to C$	3, 6 →I
1	(8)	$(A \to B) \to (A \to C)$	2, 7 →I ✦

Examples Involving &E, &I, →E, and →I

- Can you deduce the following, using &E, &I, →E, and →I?

(a) $A \to B$, $A \to C$ $\therefore A \to (B \,\&\, C)$

(b) $(A \,\&\, B) \to C$ $\therefore A \to (B \to C)$

(c) $B \,\&\, C$ $\therefore (A \to B) \,\&\, (A \to C)$

(d) $A \to B$ $\therefore (A \,\&\, C) \to (B \,\&\, C)$

(e) $A \,\&\, (B \,\&\, C)$ $\therefore A \to (B \to C)$

(f) $A \to B$ $\therefore A \to (C \to B)$

One[a] Solution to (c) (*not* solved in the text)

(c)

1	(1)	$B \,\&\, C$	Premise
2	(2)	A	Assumption
1	(3)	B	1 &E
1	(4)	C	1 &E
1	(5)	$A \to B$	2, 3 →I
1	(6)	$A \to C$	2, 4 →I
1	(7)	$(A \to B) \,\&\, (A \to C)$	5, 6 &I ✦

[a] There are many, many correct deductions of any valid argument.

One Solution to (d) (*not* in the text)

(d)

1	(1)	$A \to B$	Premise
2	(2)	$A \,\&\, C$	Assumption
2	(3)	A	2 &E
2	(4)	C	2 &E
1, 2	(5)	B	1, 3 →E
1, 2	(6)	$B \,\&\, C$	5, 4 &I
1	(7)	$(A \,\&\, C) \to (B \,\&\, C)$	2, 6 →I ✦

Important Tips For Using the →I Rule

- Use →I only when you wish to *derive* a conditional ⌜$p \to q$⌝.
- To derive ⌜$p \to q$⌝ using →I, assume the antecedent p and try to prove the consequent q. Always assume the *whole* of p, not just a part of it (like one of the conjuncts of a conjunction).
- When a conditional ⌜$p \to q$⌝ is derived by →I, the antecedent p must always be a formula which you have assumed at a previous line: it cannot be a formula that you have derived from other things. This is because it must be *discharged*.
- When you apply →I, remember to *discharge* the assumption by dropping the assumption number on the left.
- Check that the last line of your proof does not depend on any extra assumptions you have made besides your premises.