Announcements & Such

- Branden is in Chicago all of this week. He'll return next week.
- HW #2 resubmissions are due Today (4pm, drop box).
  - ☞ **Please attach your original assignment to your resub!**
  - – See my "HW Tips & Guidelines" Handout. [We're now caught-up.]
  - ☞ **Make sure you have problem #12 from p. 33 of the 4*th* printing. It's about the Mayor's election (and the council members).**
- **HW #3 has been posted. 1st-submission is due next Thursday.**
- I have posted a handout with solutions to (some of) the lecture problems on logical truth, logical falsity, equivalence, consistency, etc.
- I have also posted a handout on the "short" truth-table method, which we will be going over in lecture sometime very soon.
- Today: Chapter 3, Continued (Truth-Table methods for validity testing)

Semantic Equivalence, Contradictoriness, *etc.*: Relationships

- What are the logical relationships between $\ulcorner p$ and q are equivalent$\urcorner$, $\ulcorner p$ and q are consistent$\urcorner$, $\ulcorner p$ and q are contradictory$\urcorner$, and $\ulcorner p$ and q are inconsistent$\urcorner$? That is, which of these entails which (and which don't)?

Equivalent	Contradictory
$\Downarrow$? $\Uparrow$	$\Downarrow$? $\Uparrow$
Consistent	Inconsistent

- Answers:
  1. Equivalent $\not\Rightarrow$ Consistent (*example*?)
  2. Consistent $\not\Rightarrow$ Equivalent (*example*?)
  3. Contradictory $\Rightarrow$ Inconsistent (*why*?)
  4. Inconsistent $\not\Rightarrow$ Contradictory (*example*?)

Semantic Equivalence: Example #1

- Recall that $\ulcorner p$ unless $q\urcorner$ translates in LSL as $\ulcorner{\sim}q \to p\urcorner$.
- We've said that we can also translate $\ulcorner p$ unless $q\urcorner$ as $\ulcorner p \vee q\urcorner$.
- This is because $\ulcorner{\sim}q \to p\urcorner$ is *semantically equivalent to* $\ulcorner p \vee q\urcorner$. We may demonstrate this, using the following joint truth-table.

p	q	${\sim}q$	$\to$	p	$p \vee q$
$\top$	$\top$	$\bot$	$\top$	$\top$	$\top$
$\top$	$\bot$	$\top$	$\top$	$\top$	$\top$
$\bot$	$\top$	$\bot$	$\top$	$\bot$	$\top$
$\bot$	$\bot$	$\top$	$\bot$	$\bot$	$\bot$

- The truth-tables of $\ulcorner p \vee q\urcorner$ and $\ulcorner{\sim}q \to p\urcorner$ are the same.
- Thus, ${\sim}q \to p \mathrel{=\!\!|\!\models} p \vee q$.

Semantic Equivalence: Example #2

- $\ulcorner p \leftrightarrow q\urcorner$ is an *abbreviation* for $\ulcorner(p \to q) \,\&\, (q \to p)\urcorner$.
- The following truth-table shows it is a *legitimate* abbreviation:

p	q	$(p \to q)$	$\&$	$(q \to p)$	$p \leftrightarrow q$
$\top$	$\top$	$\top$	$\top$	$\top$	$\top$
$\top$	$\bot$	$\bot$	$\bot$	$\top$	$\bot$
$\bot$	$\top$	$\top$	$\bot$	$\bot$	$\bot$
$\bot$	$\bot$	$\top$	$\top$	$\top$	$\top$

- $\ulcorner p \leftrightarrow q\urcorner$ and $\ulcorner(p \to q) \,\&\, (q \to p)\urcorner$ have the same truth-table.
- Thus, $p \leftrightarrow q \mathrel{=\!\!|\!\models} (p \to q) \,\&\, (q \to p)$.

Semantic Equivalence: Example #3

- Intuitively, the truth-conditions for *exclusive or* ($\oplus$) are such that ⌜$p \oplus q$⌝ is true if and only if *exactly* one of p or q is true.
- I said that we could say something equivalent to this using our $\vee$, &, and $\sim$. Specifically, I said $p \oplus q \Dashv\vDash (p \vee q) \,\&\, {\sim}(p \,\&\, q)$.
- The following truth-table shows that this is correct:

p	q	$(p \vee q)$	$\&$	${\sim}(p \,\&\, q)$	$p \oplus q$
$\top$	$\top$	$\top$	$\bot$	$\bot$	$\bot$
$\top$	$\bot$	$\top$	$\top$	$\top$	$\top$
$\bot$	$\top$	$\top$	$\top$	$\top$	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\top$	$\bot$

- ⌜$p \oplus q$⌝ and ⌜$(p \vee q) \,\&\, {\sim}(p \,\&\, q)$⌝ have the same truth-table.

Some More Semantic Equivalences

- Here is a simultaneous truth-table which establishes that

$$A \leftrightarrow B \Dashv\vDash (A \,\&\, B) \vee ({\sim}A \,\&\, {\sim}B)$$

A	B	A	$\leftrightarrow$	B	$(A$	$\&$	$B)$	$\vee$	$(\sim$	A	$\&$	$\sim$	$B)$
$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\bot$	$\top$	$\bot$	$\bot$	$\top$
$\top$	$\bot$	$\top$	$\bot$	$\bot$	$\top$	$\bot$	$\bot$	$\bot$	$\bot$	$\top$	$\bot$	$\top$	$\bot$
$\bot$	$\top$	$\bot$	$\bot$	$\top$	$\bot$	$\bot$	$\top$	$\bot$	$\top$	$\bot$	$\bot$	$\bot$	$\top$
$\bot$	$\bot$	$\bot$	$\top$	$\bot$	$\bot$	$\bot$	$\bot$	$\top$	$\top$	$\bot$	$\top$	$\top$	$\bot$

- Can you prove the following equivalences with truth-tables?
  - ${\sim}(A \,\&\, B) \Dashv\vDash {\sim}A \vee {\sim}B$
  - ${\sim}(A \vee B) \Dashv\vDash {\sim}A \,\&\, {\sim}B$
  - $A \Dashv\vDash (A \,\&\, B) \vee (A \,\&\, {\sim}B)$
  - $A \Dashv\vDash A \,\&\, (B \to B)$
  - $A \Dashv\vDash A \vee (B \,\&\, {\sim}B)$

A More Complicated Equivalence (Distributivity)

- The following simultaneous truth-table establishes that

$$p \,\&\, (q \vee r) \Dashv\vDash (p \,\&\, q) \vee (p \,\&\, r)$$

p	q	r	p	$\&$	$(q \vee r)$	$(p \,\&\, q)$	$\vee$	$(p \,\&\, r)$
$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$
$\top$	$\top$	$\bot$	$\top$	$\top$	$\top$	$\top$	$\top$	$\bot$
$\top$	$\bot$	$\top$	$\top$	$\top$	$\top$	$\bot$	$\top$	$\top$
$\top$	$\bot$	$\bot$	$\top$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
$\bot$	$\top$	$\top$	$\bot$	$\bot$	$\top$	$\bot$	$\bot$	$\bot$
$\bot$	$\top$	$\bot$	$\bot$	$\bot$	$\top$	$\bot$	$\bot$	$\bot$
$\bot$	$\bot$	$\top$	$\bot$	$\bot$	$\top$	$\bot$	$\bot$	$\bot$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$

- This is *distributivity* of & over $\vee$. It also works for $\vee$ over &.

The Exhaustive Truth-Table Method for Testing Validity

- Remember, an argument is valid if it is *impossible* for its premises to be true while its conclusion is false. Let $p_1, \ldots, p_n$ be the premises of a LSL argument, and let q be the conclusion of the argument. Then, we have:

$$\begin{array}{l} p_1 \\ \vdots \\ p_n \\ \hline \therefore q \end{array}$$

is valid if and only if there is no row in the simultaneous truth-table of $p_1, \ldots, p_n$, and q which looks like the following:

atoms	premises			conclusion
$\cdots$	p_1	$\cdots$	p_n	q
$\cdots$	$\top$	$\top$	$\top$	$\bot$

- We will use simultaneous truth-tables to prove validities and invalidities. For example, consider the following valid argument:

A

$A \to B$

$\therefore B$

atoms		premises				conclusion
A	B	A	A	$\to$	B	B
$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$
$\top$	$\bot$	$\top$	$\top$	$\bot$	$\bot$	$\bot$
$\bot$	$\top$	$\bot$	$\bot$	$\top$	$\top$	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\top$	$\bot$	$\bot$

☞ VALID — there is no row in which A and $A \to B$ are both $\top$, but B is $\bot$.

- In general, we'll use the following procedure for evaluating arguments:
  1. Translate and symbolize the the argument (if given in English).
  2. Write out the symbolized argument (as above).
  3. Draw a simultaneous truth-table for the symbolized argument, outlining the columns representing the premises and conclusion.
  4. Is there a row of the table in which all premises are $\top$ but the conclusion is $\bot$? If so, the argument is invalid; if not, it's valid.
- We will practice this on examples. But, first, a "short-cut" method.

The "Short" Truth Table Method for Validity Testing I

- Consider the following LSL argument:

  $A \to (B \,\&\, E)$

  $D \to (A \vee F)$

  $\sim E$

  $\therefore D \to B$

- This argument has 3 premises and contains 5 atomic sentences. This would lead to a complete truth-table with 32 rows and 8 columns (this will be far more than 256 distinct computations).
- As such, the exhaustive truth-table method does not seem practical in this case. So, instead, let's try to construct or "reverse engineer" an invalidating interpretation.
- To do this, we "work backward" from the *assumption* that the conclusion is $\bot$ and all the premises are $\top$ on some row.

- Step 1: Assume there is an interpretation on which all three premises are $\top$ and the conclusion is $\bot$. This leads to:

A	B	D	E	F	A	$\to$	$B \,\&\, E$	D	$\to$	$(A \vee F)$	$\sim E$	$D \to B$
						$\top$			$\top$		$\top$	$\bot$

- Step 2: From the assumption that $\sim E$ is $\top$, we may infer that both E and $B \,\&\, E$ are $\bot$. This fills-in two more cells:

A	B	D	E	F	A	$\to$	$B \,\&\, E$	D	$\to$	$(A \vee F)$	$\sim E$	$D \to B$
			$\bot$			$\top$	$\bot$		$\top$		$\top$	$\bot$

- Step 3: Now, the only way that $A \to (B \,\&\, E)$ can be $\top$ (as we've assumed) is if its antecedent A is $\bot$. This yields the following:

A	B	D	E	F	A	$\to$	$B \,\&\, E$	D	$\to$	$(A \vee F)$	$\sim E$	$D \to B$
$\bot$			$\bot$		$\bot$	$\top$	$\bot$		$\top$		$\top$	$\bot$

- Step 4: Now, $D \to B$ can be $\bot$ (as we've been assuming) if and only if D is $\top$ and B is $\bot$ (just by the definition of $\to$). So:

A	B	D	E	F	A	$\to$	$B \,\&\, E$	D	$\to$	$(A \vee F)$	$\sim E$	$D \to B$
$\bot$	$\bot$	$\top$	$\bot$		$\bot$	$\top$	$\bot$	$\top$	$\top$		$\top$	$\bot$

- Step 5: Then, $D \to (A \vee F)$ can be $\top$ (as we've assumed) only if its consequent $A \vee F$ is $\top$, which gives the following:

A	B	D	E	F	A	$\to$	$B \,\&\, E$	D	$\to$	$(A \vee F)$	$\sim E$	$D \to B$
$\bot$	$\bot$	$\top$	$\bot$		$\bot$	$\top$	$\bot$	$\top$	$\top$	$\top$	$\top$	$\bot$

- Step 6: Finally, since A is $\bot$, the only way that $A \vee F$ can be $\top$ is if F is $\top$, which completes our construction!

A	B	D	E	F	A	$\to$	$B \,\&\, E$	D	$\to$	$(A \vee F)$	$\sim E$	$D \to B$
$\bot$	$\bot$	$\top$	$\bot$	$\top$	$\bot$	$\top$	$\bot$	$\top$	$\top$	$\top$	$\top$	$\bot$

The "Short" Truth Table Method for Validity Testing II

- Consider the following LSL argument:

  $\sim A \vee (B \rightarrow C)$
  $E \rightarrow (B \,\&\, A)$
  $C \rightarrow E$
  $\therefore C \leftrightarrow A$

- Let's try our "short" truth table method on this one.
- Step 1: Assume there is an interpretation on which all three premises are $\top$ and the conclusion is $\bot$. This leads to the following partial row:

A	B	C	E	$\sim A$	$\vee$	$(B \rightarrow C)$	E	$\rightarrow$	$(B \,\&\, A)$	$C \rightarrow E$	$C \leftrightarrow A$
					$\top$			$\top$		$\top$	$\bot$

- Step 2: Now, there are *two* ways the conclusion $C \leftrightarrow A$ can be false:
  - Case 1: C is $\top$ and A is $\bot$.
  - Case 2: C is $\bot$ and A is $\top$.

- Step 2 (Case 1): C is $\top$ and A is $\bot$. This leads to the following:

A	B	C	E	$\sim A$	$\vee$	$(B \rightarrow C)$	E	$\rightarrow$	$(B \,\&\, A)$	$C \rightarrow E$	$C \leftrightarrow A$
$\bot$		$\top$		$\top$	$\top$			$\top$	$\bot$	$\top$	$\bot$

- Step 3 (Case 1): Now, the *only* way to make $E \rightarrow (B \,\&\, A)$ T is to make E $\bot$. But, this *contradicts* constraints already forced on our construction!

A	B	C	E	$\sim A$	$\vee$	$(B \rightarrow C)$	E	$\rightarrow$	$(B \,\&\, A)$	$C \rightarrow E$	$C \leftrightarrow A$
$\bot$		$\top$	$\bot$	$\top$	$\top$		$\bot$	$\top$	$\bot$	$\top / \bot$!!	$\bot$

  - All this shows (so far) is that there is no row in which all the premises are $\top$ and the conclusion is $\bot$ *by way of C being $\top$ and A being $\bot$*. But, this does not yet show that the argument is valid! We must also check Case 2.

- Step 2 (Case 2): C is $\bot$ and A is $\top$. This leads to the following, initially:

A	B	C	E	$\sim A$	$\vee$	$(B \rightarrow C)$	E	$\rightarrow$	$(B \,\&\, A)$	$C \rightarrow E$	$C \leftrightarrow A$
$\top$		$\bot$			$\top$			$\top$		$\top$	$\bot$

- Step 3 (Case 2): If A is $\top$, then $\sim A$ is $\bot$. So, making $\sim A \vee (B \rightarrow C)$ $\top$ will require making $B \rightarrow C$ $\top$, which implies that B is $\bot$:

A	B	C	E	$\sim A$	$\vee$	$(B \rightarrow C)$	E	$\rightarrow$	$(B \,\&\, A)$	$C \rightarrow E$	$C \leftrightarrow A$
$\top$	$\bot$	$\bot$		$\bot$	$\top$	$\top$		$\top$		$\top$	$\bot$

- Step 4 (Case 2): Since B must be $\bot$, so must $B \,\&\, A$. As a result, the only way to make $E \rightarrow (B \,\&\, A)$ $\top$ is to make E $\bot$. Success!!

A	B	C	E	$\sim A$	$\vee$	$(B \rightarrow C)$	E	$\rightarrow$	$(B \,\&\, A)$	$C \rightarrow E$	$C \leftrightarrow A$
$\top$	$\bot$	$\bot$	$\bot$	$\bot$	$\top$	$\top$	$\bot$	$\top$	$\bot$	$\top$	$\bot$

  - So, we have found a row in which all premises are $\top$ and the conclusion is $\bot$. Thus, the argument is *invalid* after all!
  - The moral of this example is that we must exhaust all possible ways of constructing an invalidating row, until we either find one (invalid) or we have shown that all possible constructions lead to contradiction (valid).
  - Let's look at a case of a *valid* argument.

The "Short" Truth Table Method for Validity Testing III

- Consider the following LSL argument:

  $(A \,\&\, B) \vee (A \,\&\, C)$
  $\therefore A \,\&\, (B \vee C)$

- Let's try our "short" truth table method on this one.
- Step 1: Assume there is an interpretation on which the premises is $\top$ but the conclusion is $\bot$. This leads to the following partial row:

A	B	C	$(A \,\&\, B)$	$\vee$	$(A \,\&\, C)$	A	$\&$	$(B \vee C)$
				$\top$			$\bot$	

- Already, we have to break this down into cases, since there are three ways the premise can be $\top$ and also three ways the conclusion can be $\bot$.
  - Case 1: A is $\bot$ and $B \vee C$ is $\bot$.
  - Case 2: A is $\top$ and $B \vee C$ is $\bot$.
  - Case 3: A is $\bot$ and $B \vee C$ is $\top$.

- Step 2 (Case 1): A is $\bot$ and $B \vee C$ is $\bot$. This leads to the following:

A	B	C	$(A \& B)$	$\vee$	$(A \& C)$	A	$\&$	$(B \vee C)$
$\bot$				$\top$		$\bot$	$\bot$	$\bot$

- Step 3 (Case 1): If A is $\bot$, then so are both conjunctions $A \& B$ and $A \& C$, which contradicts our assumption that the premise is $\top$! Dead end.

A	B	C	$(A \& B)$	$\vee$	$(A \& C)$	A	$\&$	$(B \vee C)$
$\bot$			$\bot$	$\top/\bot$!!	$\bot$	$\bot$	$\bot$	$\bot$

  - So, Case 1 contains no counterexamples to the validity of the argument. This does not imply that the argument is valid! We must check *all* cases before we can conclude that an argument is valid. On to Case 2...

- Step 2 (Case 2): A is $\top$ and $B \vee C$ is $\bot$. This leads to the following:

A	B	C	$(A \& B)$	$\vee$	$(A \& C)$	A	$\&$	$(B \vee C)$
$\top$				$\top$		$\top$	$\bot$	$\bot$

- Step 3 (Case 2): If $B \vee C$ is $\bot$, then both B and C are $\bot$, which leads to both conjunctions $A \& B$ and $A \& C$ being $\bot$ — contradiction — another dead end.

A	B	C	$(A \& B)$	$\vee$	$(A \& C)$	A	$\&$	$(B \vee C)$
$\top$	$\bot$	$\bot$	$\bot$	$\top/\bot$!!	$\bot$	$\top$	$\bot$	$\bot$

  - We cannot yet conclude that the argument is valid! On to Case 3...

- Step 2 (Case 3): A is $\bot$ and $B \vee C$ is $\top$. This leads to the following:

A	B	C	$(A \& B)$	$\vee$	$(A \& C)$	A	$\&$	$(B \vee C)$
$\bot$				$\top$		$\bot$	$\bot$	$\top$

- Step 3 (Case 3): If A is $\bot$, then so are both conjunctions $A \& B$ and $A \& C$, which contradicts our assumption that the premise is $\top$! Dead end.

A	B	C	$(A \& B)$	$\vee$	$(A \& C)$	A	$\&$	$(B \vee C)$
$\bot$			$\bot$	$\top/\bot$!!	$\bot$	$\bot$	$\bot$	$\bot$

  - Finally, we see the argument is valid (no cases yield counterexamples).

The "Short" Method for Constructing Interpretations: Handout Problem #1

- Question: $A \to (C \vee E), B \to D \overset{?}{\vDash} (A \vee B) \to (C \to (D \vee E))$.
- Answer: $A \to (C \vee E), B \to D \nvDash (A \vee B) \to (C \to (D \vee E))$.
- Step 1: Assume there is an interpretation on which the premises is $\top$ but the conclusion is $\bot$. This leads to the following partial row:

A	B	C	D	E	$A \to (C \vee E)$	$B \to D$	$(A \vee B) \to (C \to (D \vee E))$
					$\top$	$\top$	$\bot$

- Step 2: There's only one way the conclusion can be $\bot$, which leads to:

A	B	C	D	E	$A \to (C \vee E)$	$B \to D$	$(A \vee B) \to (C \to (D \vee E))$
					$\top$	$\top$	$\top$ $\bot$ $\bot$

- Step 3: There's only one way $C \to (D \vee E)$ can be $\bot$, which leads to:

A	B	C	D	E	$A \to (C \vee E)$	$B \to D$	$(A \vee B) \to (C \to (D \vee E))$
		$\top$			$\top$	$\top$	$\top$ $\bot$ $\top$ $\bot$ $\bot$

- Step 4: There's only one way $D \vee E$ can be $\bot$, which leads to:

A	B	C	D	E	$A \to (C \vee E)$	$B \to D$	$(A \vee B) \to (C \to (D \vee E))$
		$\top$	$\bot$	$\bot$	$\top$	$\top$ $\bot$	$\top$ $\bot$ $\top$ $\bot$ $\bot$

- Step 5: Since D is $\bot$, the only way $B \to D$ can be $\top$ is if B is $\bot$:

A	B	C	D	E	$A \to (C \vee E)$	$B \to D$	$(A \vee B) \to (C \to (D \vee E))$
	$\bot$	$\top$	$\bot$	$\bot$	$\top$	$\bot$ $\top$ $\bot$	$\top$ $\bot$ $\top$ $\bot$ $\bot$

- Step 6: Since C is $\top$, so is $C \vee E$, which makes $A \to (C \vee E)$ $\top$ *regardless of the truth-value of* A. So, I will just let A be $\top$, and then we're done.

A	B	C	D	E	$A \to (C \vee E)$	$B \to D$	$(A \vee B) \to (C \to (D \vee E))$
$\top$	$\bot$	$\top$	$\bot$	$\bot$	$\top$ $\top$ $\top$	$\bot$ $\top$ $\bot$	$\top$ $\bot$ $\top$ $\bot$ $\bot$

- When reporting your answer, all you need to do is give the single row that serves as a counterexample. Here, I recommend you include the quasi-columns that you used to calculate the truth-values in the row.
- Verbal explanations are optional. Here's the detailed handout solution.

Answer. $A \to (C \vee E), B \to D \nvDash (A \vee B) \to (C \to (D \vee E))$

Explanation.[a] Assume that '$A \to (C \vee E)$' is $\top$, '$B \to D$' is $\top$, and '$(A \vee B) \to (C \to (D \vee E))$' is $\bot$. In order for '$(A \vee B) \to (C \to (D \vee E))$' to be $\bot$, both '$A \vee B$' and 'C' must be $\top$, and both 'D' and 'E' must be $\bot$. This *guarantees* that the first premise is $\top$ (since '$A \to (C \vee E)$' *must*, at this point, have a $\top$ consequent). We can also make the second premise $\top$, simply by making 'B' $\bot$. So, as the following single-row truth-table shows, we have *succeeded* in finding an interpretation on which '$A \to (C \vee E)$' and '$B \to D$' are both $\top$, but '$(A \vee B) \to (C \to (D \vee E))$' is $\bot$. *QED*.

A	B	C	D	E	$A \to (C \vee E)$	$B \to D$	$(A \vee B) \to (C \to (D \vee E))$
$\top$	$\bot$	$\top$	$\bot$	$\bot$	$\top\ \top\ \top\ \top\ \bot$	$\bot\ \top\ \bot$	$\top\ \top\ \bot\ \bot\ \top\ \bot\ \bot\ \bot\ \bot$

[a]You do *not* have to show *all* of your reasoning in cases like this one, where the argument is *in*valid (*i.e.*, where $\nvDash$). I am just showing you *all* of *my* reasoning to give you more information about how these kinds of problems are solved. All you *need* to do here is report an interpretation (*i.e.*, a single-row) which invalidates the inference. But, when you do so, I recommend filling-in all of the quasi-columns to make explicit all of the calculations required.

The "Short" Method for Constructing Interpretations: Handout Problem #2

- Question: $A \leftrightarrow (B \vee C), B \to D, D \leftrightarrow C \overset{?}{\vDash} A \leftrightarrow D$.
- Answer: $A \leftrightarrow (B \vee C), B \to D, D \leftrightarrow C \vDash A \leftrightarrow D$.
- Step 1: Assume there is an interpretation on which the premises is $\top$ but the conclusion is $\bot$. This leads to the following partial row:

A	B	C	D	$A \leftrightarrow (B \vee C)$	$B \to D$	$D \leftrightarrow C$	$A \leftrightarrow D$
				$\top$	$\top$	$\top$	$\bot$

- Already, we have to break this down into cases, since there are ($\geq$) two ways each premise can be $\top$ and also two ways the conclusion can be $\bot$.
  - Case 1: A is $\top$ and D is $\bot$.
  - Case 2: A is $\bot$ and D is $\top$.

- Step 2 (Case 1): If A is $\top$ and D is $\bot$, then we have the following:

A	B	C	D	$A \leftrightarrow (B \vee C)$	$B \to D$	$D \leftrightarrow C$	$A \leftrightarrow D$
$\top$			$\bot$	$\top\ \top$	$\top\ \bot$	$\bot\ \top$	$\top\ \bot\ \bot$

- Step 3 (Case 1): Now, the only way for $B \to D$ to be $\top$ is for B to be $\bot$. And, the only way for $D \leftrightarrow C$ to be $\top$ is for C to be $\bot$, which yields:

A	B	C	D	$A \leftrightarrow (B \vee C)$	$B \to D$	$D \leftrightarrow C$	$A \leftrightarrow D$
$\top$	$\bot$	$\bot$	$\bot$	$\top\ \top\ \bot\ \ \bot$	$\bot\ \top\ \bot$	$\bot\ \top\ \bot$	$\top\ \bot\ \bot$

- Step 4 (Case 1): But, we need $A \leftrightarrow (B \vee C)$ to be $\top$, which means we need $B \vee C$ to be $\top$. However, this contradicts our assumptions — dead end!

A	B	C	D	$A \leftrightarrow (B \vee C)$	$B \to D$	$D \leftrightarrow C$	$A \leftrightarrow D$
$\top$	$\bot$	$\bot$	$\bot$	$\top\ \top\ \bot\ \top/\bot!!\ \bot$	$\bot\ \top\ \bot$	$\bot\ \top\ \bot$	$\top\ \bot\ \bot$

- As usual, we cannot infer — yet — that this argument is valid.
- We must continue on with an examination of Case 2 . . .

- Step 2 (Case 2): If A is $\bot$ and D is $\top$, then we have the following:

A	B	C	D	$A \leftrightarrow (B \vee C)$	$B \to D$	$D \leftrightarrow C$	$A \leftrightarrow D$
$\bot$			$\top$	$\bot\ \top$	$\top\ \top$	$\top\ \top$	$\bot\ \bot\ \top$

- Step 3 (Case 2): Now, the only way for $D \leftrightarrow C$ to be $\top$ is for C to be $\top$, which forces $B \vee C$ to be $\top$, contradicting our assumptions — dead end!

A	B	C	D	$A \leftrightarrow (B \vee C)$	$B \to D$	$D \leftrightarrow C$	$A \leftrightarrow D$
$\bot$		$\top$	$\top$	$\bot\ \top/\bot!!\ \ \top\ \top$	$\top\ \top$	$\top\ \top\ \top$	$\bot\ \bot\ \top$

- Since *both* of the two possible cases lead to a dead-end (*i.e.*, a contradiction), we may (finally) infer that this argument is *valid*.
- For valid arguments, you must give a verbal explanation of your "short" method answers. The handout contains two model solutions.
- Here's what the model solution on the handout looks like for this problem. Note: there are no "partial rows" included in the solution. You *may* include these (as in the lecture notes above), but you *need not*.

Answer. $A \leftrightarrow (B \vee C), B \rightarrow D, D \leftrightarrow C \vDash A \leftrightarrow D$.

Explanation. Assume '$A \leftrightarrow (B \vee C)$' is $\top$, '$B \rightarrow D$' is $\top$, '$D \leftrightarrow C$' is $\top$, and '$A \leftrightarrow D$' is $\bot$. There are *exactly two* ways in which '$A \leftrightarrow D$' can be $\bot$:

1. 'A' is $\top$, and 'D' is $\bot$. In this case, in order for '$D \leftrightarrow C$' to be $\top$, 'C' must be $\bot$. And, in order for '$B \rightarrow D$' to be $\top$, 'B' must be $\bot$. This means that the *disjunction* '$B \vee C$' must be $\bot$. So, in order for '$A \leftrightarrow (B \vee C)$' to be $\top$, we must have 'A' $\bot$ as well, which contradicts our assumption. So, in this first case, we have been forced into a *contradiction*.
2. 'A' is $\bot$, and 'D' is $\top$. In this case, in order for '$D \leftrightarrow C$' to be $\top$, 'C' must be $\top$. But, if 'C' is $\top$, then so is '$B \vee C$'. Hence, if '$A \leftrightarrow (B \vee C)$' is going to be $\top$, then 'A' must be $\top$, which contradicts our assumption. So, in this second (and *last*) case, we have been forced into a *contradiction*.

$\therefore$ There are no interpretations on which '$A \leftrightarrow (B \vee C)$', '$B \rightarrow D$', and '$D \leftrightarrow C$' are all $\top$ and '$A \leftrightarrow D$' is $\bot$. So, $A \leftrightarrow (B \vee C), B \rightarrow D, D \leftrightarrow C \vDash A \leftrightarrow D$. $\square$

Presenting Your "Short-Cut" Truth-Table Tests

- In any application of the "short" method, there are two possibilities:

1. You find an interpretation (*i.e.*, a row of the truth-table) on which all the premises $p_1, \ldots, p_n$ of an argument are true and the conclusion q is false. *All you need to do here* is (*i*) write down the relevant row of the truth-table, and (*ii*) say "Here is an interpretation on which $p_1, \ldots, p_n$ are all true and q is false. So, $p_1, \ldots, p_n \therefore q$ is *in*valid."
2. You discover that there is *no possible way* of making $p_1, \ldots, p_n$ true and q false. Here, you need to *explain all of your reasoning* (as I do in lecture, or as Forbes does, or as I do in my handout). It must be clear that you have *exhausted all possible cases*, before concluding that $p_1, \ldots, p_n \therefore q$ is *valid*. This can be rather involved, and should be spelled out in a step-by-step fashion. Each salient case has to be examined.

- Consult my handout and lecture notes for model answers of both kinds.