Working with LMPL Interpretations

Philosophy 12A
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1 Working with *Given* LMPL Interpretations

Consider the following (*given*) LMPL interpretation $\mathcal{I}_1$:

($\mathcal{I}_1$)

	F	G	H	I	J
α	+	+	−	+	−
β	−	−	−	+	+
γ	+	−	−	−	+

In other words, the interpretation $\mathcal{I}_1$ has the following features: $\mathcal{D} = \{\alpha, \beta, \gamma\}$, $\text{Ext}(F) = \{\alpha, \gamma\}$, $\text{Ext}(G) = \{\alpha\}$, $\text{Ext}(H) = \varnothing$ (where, $\varnothing$ is *the null set*), $\text{Ext}(I) = \{\alpha, \beta\}$, and $\text{Ext}(J) = \{\beta, \gamma\}$.

Question: What are the $\mathcal{I}_1$-truth-values of ①–⑥?

① $\sim Ja$
② $Fc \to Ic$
③ $(\exists x)(Jx \leftrightarrow Hx)$
④ $(\forall x)[Jx \to (Gx \vee Fx)]$
⑤ $(\exists x)Gx \to (\forall y)(Fy \vee Gy)$
⑥ $(\exists y)(\forall x)[Gy \,\&\, (Jx \to (Ix \vee Fx))]$

Solutions:

① '$\sim Ja$' is *true* on $\mathcal{I}_1$. This is because 'Ja' is *false* on $\mathcal{I}_1$, since $\alpha \notin \text{Ext}(J)$.

② '$Fc \to Ic$' is *false* on $\mathcal{I}_1$. This is because its *antecedent* 'Fc' is *true* on $\mathcal{I}_1$, since $\gamma \in \text{Ext}(F)$; but its *consequent* 'Ic' is *false* on $\mathcal{I}_1$, since $\gamma \notin \text{Ext}(I)$.

③ '$(\exists x)(Jx \leftrightarrow Hx)$' is *true* on $\mathcal{I}_1$. The *instances* of '$(\exists x)(Jx \leftrightarrow Hx)$' on $\mathcal{I}_1$ are: (*i*) '$Ja \leftrightarrow Ha$', (*ii*) '$Jb \leftrightarrow Hb$', and (*iii*) '$Jc \leftrightarrow Hc$'. Instances (*ii*) and (*iii*) are *false* on $\mathcal{I}_1$ (*why*?). But, instance (*i*) is *true* on $\mathcal{I}_1$, because 'Ja' and 'Ha' are *both false* on $\mathcal{I}_1$, since $\alpha \notin \text{Ext}(J)$ and $\alpha \notin \text{Ext}(H)$.[1]

④ '$(\forall x)[Jx \to (Gx \vee Fx)]$' is *false* on $\mathcal{I}_1$. The *instances* of '$(\forall x)[Jx \to (Gx \vee Fx)]$' on $\mathcal{I}_1$ are as follows: (*i*) '$Ja \to (Ga \vee Fa)$', (*ii*) '$Jb \to (Gb \vee Fb)$', and (*iii*) '$Jc \to (Gc \vee Fc)$'. Instances (*i*) and (*iii*) are *true* on $\mathcal{I}_1$ (*why*?). But, instance (*ii*) is *false* on $\mathcal{I}_1$. This is because 'Jb' is *true* on $\mathcal{I}_1$, since $\beta \in \text{Ext}(J)$; but '$Gb \vee Fb$' is *false* on $\mathcal{I}_1$, since $\beta \notin \text{Ext}(G)$ and $\beta \notin \text{Ext}(F)$.[2]

⑤ '$(\exists x)Gx \to (\forall y)(Fy \vee Gy)$' is $\perp$ on $\mathcal{I}_1$. The *antecedent* '$(\exists x)Gx$' of this conditional is $\top$ on $\mathcal{I}_1$, because its *instance* 'Ga' is $\top$ on $\mathcal{I}_1$, since $\alpha \in \text{Ext}(G)$. But, the *consequent* '$(\forall y)(Fy \vee Gy)$' of this conditional is *false* on $\mathcal{I}_1$, because its *instance* '$Fb \vee Gb$' is false on $\mathcal{I}_1$, since $\beta \notin \text{Ext}(G)$ and $\beta \notin \text{Ext}(F)$.

⑥ '$(\exists y)(\forall x)[Gy \,\&\, (Jx \to (Ix \vee Fx))]$' is *true* on $\mathcal{I}_1$. The three *instances* of ⑥ on $\mathcal{I}_1$ are as follows:

(*i*) '$(\forall x)[Ga \,\&\, (Jx \to (Ix \vee Fx))]$'. This instance of ⑥ is $\top$ on $\mathcal{I}_1$. The *instances* of (*i*) are as follows: (*i*.1) '$Ga \,\&\, (Ja \to (Ia \vee Fa))$', (*i*.2) '$Ga \,\&\, (Jb \to (Ib \vee Fb))$', and (*i*.3) '$Ga \,\&\, (Jc \to (Ic \vee Fc))$'. (*i*.1) is $\top$ on $\mathcal{I}_1$ since *both* 'Ga' [$\alpha \in \text{Ext}(G)$], *and* '$Ja \to (Ia \vee Fa)$' [$\alpha \notin \text{Ext}(J)$] are $\top$ on $\mathcal{I}_1$. (*i*.2) is $\top$ on $\mathcal{I}_1$ since *both* 'Ga' *and* '$Jb \to (Ib \vee Fb)$' [$\beta \in \text{Ext}(J)$ and $\beta \in \text{Ext}(I)$] are $\top$ on $\mathcal{I}_1$. (*i*.3) is $\top$ on $\mathcal{I}_1$ since *both* 'Ga' *and* '$Jc \to (Ic \vee Fc)$' [$\gamma \in \text{Ext}(J)$ and $\gamma \in \text{Ext}(F)$] are $\top$ on $\mathcal{I}_1$.

(*ii*) '$(\forall x)[Gb \,\&\, (Jx \to (Ix \vee Fx))]$'. This instance of ⑥ is $\perp$ on $\mathcal{I}_1$, because 'Gb' is false on $\mathcal{I}_1$, since $\beta \notin \text{Ext}(G)$. So, *none* of the three instances of '$(\forall x)[Gb \,\&\, (Jx \to (Ix \vee Fx))]$' is true on $\mathcal{I}_1$ (*why*?).

(*iii*) '$(\forall x)[Gc \,\&\, (Jx \to (Ix \vee Fx))]$'. This instance of ⑥ is $\perp$ on $\mathcal{I}_1$, because 'Gc' is false on $\mathcal{I}_1$, since $\gamma \notin \text{Ext}(G)$. So, *none* of the three instances of '$(\forall x)[Gc \,\&\, (Jx \to (Ix \vee Fx))]$' is true on $\mathcal{I}_1$ (*why*?).

☞ *All* instances of (*i*) are $\top$ on $\mathcal{I}_1$. $\therefore$ (*i*) is $\top$ on $\mathcal{I}_1$. $\therefore$ *One* instance of ⑥ is true on $\mathcal{I}_1$. $\therefore$ ⑥ is $\top$ on $\mathcal{I}_1$.

[1] Remember, it only takes *one true instance* (on $\mathcal{I}$) of the *existential* claim $\ulcorner(\exists v)\phi v\urcorner$ to make $\ulcorner(\exists v)\phi v\urcorner$ *true* on $\mathcal{I}$.
[2] Remember, it only takes *one false instance* (on $\mathcal{I}$) of the *universal* claim $\ulcorner(\forall v)\phi v\urcorner$ to make $\ulcorner(\forall v)\phi v\urcorner$ *false* on $\mathcal{I}$.

2 *Constructing* LMPL Interpretations to Prove $\nvDash$ Claims

Problem #1. Show that:

(1) $(\forall x)(Fx \to Gx), (\forall x)(Fx \to Hx) \nvDash (\forall x)(Gx \to Hx)$.

Solution. In order to prove (1), we need to construct an interpretation $\mathcal{I}$ on which '$(\forall x)(Fx \to Gx)$' and '$(\forall x)(Fx \to Hx)$' are both true, but '$(\forall x)(Gx \to Hx)$' is false. We proceed in several steps.

- **Step 1**: We begin — *provisionally* — with the smallest possible domain $\mathcal{D} = \{\alpha\}$.
- **Step 2**: We make sure that the object α is a *counterexample* to the conclusion '$(\forall x)(Gx \to Hx)$'. That is, we make sure that the *instance* '$Ga \to Ha$' of the conclusion is *false* on $\mathcal{I}$. So, we must have $\alpha \in \text{Ext}(G)$, but $\alpha \notin \text{Ext}(H)$. We can achieve this by making $\text{Ext}(G) = \{\alpha\}$, and $\text{Ext}(H) = \varnothing$.
- **Step 3**: At the same time, we try to make *both* of the premises '$(\forall x)(Fx \to Gx)$' *and* '$(\forall x)(Fx \to Hx)$' *true* on $\mathcal{I}$. In this case, we can make both premises true simply by ensuring that $\alpha \notin \text{Ext}(F)$. The simplest way to do this is to stipulate that $\text{Ext}(F) = \varnothing$ — which yields the following interpretation:

($\mathcal{I}_2$)

	F	G	H
α	$-$	$+$	$-$

We have discovered an interpretation $\mathcal{I}_2$ on which '$(\forall x)(Fx \to Gx)$' and '$(\forall x)(Fx \to Hx)$' are both true, but '$(\forall x)(Gx \to Hx)$' is false (*demonstrate this*!). Therefore, claim (1) is true.[3]

Problem #2. Show that:

(2) $(\exists x)(Fx \,\&\, Gx), (\exists x)(Fx \,\&\, Hx), (\forall x)(Gx \to {\sim}Hx) \nvDash (\forall x)[Fx \leftrightarrow (Gx \vee Hx)]$.

Solution. In order to prove (2), we need to construct an interpretation $\mathcal{I}$ on which '$(\exists x)(Fx\&Gx)$', '$(\exists x)(Fx\& Hx)$', and '$(\forall x)(Gx \to {\sim}Hx)$' are all true, but '$(\forall x)[Fx \leftrightarrow (Gx \vee Hx)]$' is false.

- **Step 1**: We begin — *provisionally* — with the smallest possible domain $\mathcal{D} = \{\alpha\}$.
- **Step 2**: We make sure that the object α is a *counterexample* to the conclusion '$(\forall x)[Fx \leftrightarrow (Gx \vee Hx)]$'. So, we make its *instance* '$Fa \leftrightarrow (Ga \vee Ha)$' *false* on $\mathcal{I}$. There are several ways to do this. One way: $\alpha \in \text{Ext}(F)$, $\alpha \notin \text{Ext}(G)$, and $\alpha \notin \text{Ext}(H)$. So far, we have $\text{Ext}(F) = \{\alpha\}$, and $\text{Ext}(G) = \text{Ext}(H) = \varnothing$.
- **Step 3**: Now, we must try to make *all three* of the premises (*i*) '$(\exists x)(Fx \,\&\, Gx)$', (*ii*) '$(\exists x)(Fx \,\&\, Hx)$', and (*iii*) '$(\forall x)(Gx \to {\sim}Hx)$' *true* on $\mathcal{I}$. In order to make (*i*) true on $\mathcal{I}$, we must ensure that there is some object in the domain $\mathcal{D}$ which satisfies *both* predicates 'F' *and* 'G'. But, since α must *not* satisfy both 'F' and 'G', this means we will need to *add another object* β to our domain $\mathcal{D}$, such that: $\beta \in \text{Ext}(F)$, and $\beta \in \text{Ext}(G)$. Now, we have $\text{Ext}(F) = \{\alpha, \beta\}$, $\text{Ext}(G) = \{\beta\}$, and $\text{Ext}(H) = \varnothing$. All that remains is to ensure that premises (*ii*) and (*iii*) are also true on $\mathcal{I}$. In order to make (*ii*) true on $\mathcal{I}$, we'll need to make sure that there is some object in $\mathcal{D}$ which satisfies *both* predicates 'F' *and* 'H'. We could *try* to make β satisfy *all three* predicates 'F', 'G', *and* 'H'. But, if we were to do this, then premise (*iii*) would become *false* on $\mathcal{I}$, since its *instance* '$Gb \to {\sim}Hb$' would then be false on $\mathcal{I}$. Thus, we'll need to *add a third object* γ to $\mathcal{D}$ such that: $\gamma \in \text{Ext}(F)$, $\gamma \notin \text{Ext}(G)$, and $\gamma \in \text{Ext}(H)$ — success:

($\mathcal{I}_3$)

	F	G	H
α	$+$	$-$	$-$
β	$+$	$+$	$-$
γ	$+$	$-$	$+$

We have discovered an interpretation $\mathcal{I}_3$ on which '$(\exists x)(Fx\&Gx)$', '$(\exists x)(Fx\&Hx)$', and '$(\forall x)(Gx \to {\sim}Hx)$' are all true, but '$(\forall x)[Fx \leftrightarrow (Gx \vee Hx)]$' is false (*demonstrate this*!). Therefore, claim (2) is true.

[3]When you're asked to prove a claim like (1), you must do *two* things: (*i*) *Report* an interpretation (like $\mathcal{I}_2$) which serves as a counterexample to the validity of the LMPL argument-form, *and* (*ii*) *Demonstrate* that your interpretation *really is* a counterexample — *i.e.*, *show* that your interpretation makes all the premises true and the conclusion false, using the methods on the front page of this handout. You do ***not*** need to explain the process which led to the *discovery* of the interpretation.