Announcements and Such

- Today's Music: *The Doors*
- I have posted my solutions to HW #4 and HW #5.
- HW #6 is due on Thursday @ 4pm.

☞ **The final is *in class* on Thursday. You'll be given 3 hours to do it.**

- I've posted two important handouts concerning the final exam:
  - The (Complete) Natural Deduction Rules Handout (provided at final).
  - A sample final exam, which has the same structure as the actual final. This sample was discussed, in detail, in lecture last week.
- Today: Chapters 7 & 8 — L2PL
  - I will only be covering (some of) the **L2PL** parts of Chapters 7 & 8.
  - I will say something about what will be on the final at the end of today's lecture (which, alas, will be my last lecture at Berkeley).

Why (†) is So Important — L2PL *vs* LMPL: Infinite Domains

- In LMPL, if p is true on any interpretation $\mathcal{I}$, then it is true on a *finite* interpretation. Indeed, p will be true on an interpretation of size no greater than 2^k, where k is the # of monadic predicate letters in p.
- In L2PL, some statements are true *only* on *infinite* interpretations. It is for this reason that there is no general decision procedure for validity (or logical truth) in L2PL. (†) on the last slide is a good example of this.

(†) $(\forall x)(\exists y)Rxy, (\forall x)(\forall y)(\forall z)[(Rxy \,\&\, Ryz) \to Rxz] \nvDash (\exists x)Rxx$

- **Fact**. *Only infinite interpretations $\mathcal{I}$ can be counterexamples to the validity in* (†). To see why, try to *construct* such an interpretation.
- We start by showing that no interpretation $\mathcal{I}_1$ with a 1-element domain can be an interpretation on which the premises of (†) are $\top$ and its conclusion is $\bot$. Then, we will repeat this argument for I_2 and $\mathcal{I}_3$.
- This reasoning can, in fact, be shown correct for *all* (finite) n. So, only $\mathcal{I}$'s with infinite domains will work [*e.g.*, $\mathcal{D} = \mathbb{N}$, Rxy: $x < y$].
- Begin with a 1-element domain $\{\alpha\}$. For the conclusion of (4) to be $\bot$, no

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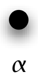
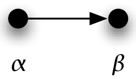
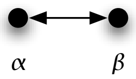
object can be related to itself: $(\forall x){\sim}Rxx$. Thus, we must have ${\sim}Raa$:



- But, to make the first premise $\top$, we need there to be *some* y such that Ray is $\top$. That means we need *another object* β to allow Rab. Thus:

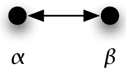


- Now, because we need the conclusion to remain $\bot$, we must have ${\sim}Rbb$. And, because we need the first premise to remain $\top$, we need there to be *some* y such that Rby is $\top$. We could *try* to make Rba $\top$, as follows:

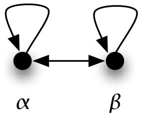


- But, this picture is not consistent with the second premise being $\top$ and (at the same time) the conclusion being $\bot$. If R is transitive, then $Rab \,\&\, Rba$ (as pictured) entails Raa, which makes the conclusion $\top$.

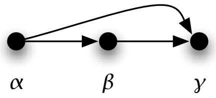
Transitivity of and



entails:

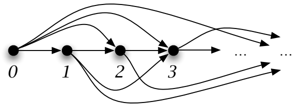


- Thus, the only way to consistently ensure that there is some y such that Rby is to introduce *yet another object* γ (such that Rbc), which yields:



- Again, in order to make the conclusion $\bot$, we must have ${\sim}Rcc$, and in order to make the first premise $\top$, there must be some y such that Rcy.
- We could *try* to make either Rca or Rcb true. But, both of these choices will end-up with the same sort of inconsistency we just saw with β.

- In other words, *no finite interpretation* will give us what we want here.
- However, if we let $\mathcal{D} = \mathbb{N}$ and Rxy: $x < y$, then we get what we want.



- That is, the relation Rxy: $x < y$ *on the natural numbers* $\mathbb{N}$ is such that:
  - For all x, there exists a y such that $x < y$. [seriality]
  - For all x, y, z, if $x < y$ and $y < z$, then $x < z$. [transitivity]
  - For all x, $x \not< x$. [irreflexivity]
- It is crucial that the set $\mathbb{N}$ of *all* natural numbers is *infinite*. The relation $<$ cannot satisfy all three of these properties on *any finite* domain.
- *I.e.*, no finite subset of $\mathbb{N}$ will suffice to show that the invalidity in (4) holds. Equivalently, the following sentence of L2PL is $\perp$ on *all finite* $\mathcal{I}$'s:

  $p \overset{\text{def}}{=} (\forall x)(\exists y)Rxy \,\&\, (\forall x)(\forall y)(\forall z)[(Rxy \,\&\, Ryz) \to Rxz] \,\&\, (\forall x){\sim}Rxx$
- This sort of thing *cannot happen* in LMPL. In this sense, the introduction of a single 2-place predicate involves a *quantum leap* in complexity.

Some Further Remarks on Validity in L2PL

- As I just explained, there is no general decision procedure for $\vDash$ claims in L2PL. This is because we can't always establish $\nvDash$ claims in finite time.
- However, there is a method for proving $\vDash$ claims — *natural deduction*. And, L2PL's natural deduction system *is exactly the same as LMPL's*!
- Before we get to proofs, however, I want to look at the alternating quantifier example that I said separates LMPL and L2PL.
- As we have seen, $(\forall x)(\exists y)Rxy \nvDash (\exists y)(\forall x)Rxy$. But, the converse entailment *does* hold. That is, $(\exists y)(\forall x)Rxy \vDash (\forall x)(\exists y)Rxy$.
- We will *prove — i.e., deduce —* $(\exists y)(\forall x)Rxy \vdash (\forall x)(\exists y)Rxy$ shortly.
- Before we do that, let's think about $(\exists y)(\forall x)Rxy \vDash (\forall x)(\exists y)Rxy$ using our definitions, and our informal method of thinking of $\forall$ as & and $\exists$ as $\vee$. This is interesting for both directions of the entailment.
- But, we need to be much more careful here than with LMPL!

- First, consider what $(\exists y)(\forall x)Rxy$ says on a domain of size n:

  $$(\exists y)(\forall x)Rxy \approx_n (\forall x)Rxa \vee (\forall x)Rxb \vee \cdots \vee (\forall x)Rxn$$
  $$\approx_n (Raa \,\&\, \cdots \,\&\, Rna) \vee (Rab \,\&\, \cdots \,\&\, Rnb) \vee \cdots \vee (Ran \,\&\, \cdots \,\&\, Rnn)$$
- Next, consider what $(\forall x)(\exists y)Rxy$ says on a domain of size n:

  $$(\forall x)(\exists y)Rxy \approx_n (\exists y)Ray \,\&\, (\exists y)Rby \,\&\, \cdots \,\&\, (\exists y)Rny$$
  $$\approx_n (Raa \vee \cdots \vee Ran) \,\&\, (Rba \vee \cdots \vee Rbn) \,\&\, \cdots \,\&\, (Rna \vee \cdots \vee Rnn)$$
- Then, we notice that these two sentential forms are intimately related. Specifically, we note that $(\exists y)(\forall x)Rxy$ has the following n-form:

  $\mathcal{X}_n = (p_1 \,\&\, p_2 \,\&\, \cdots \,\&\, p_n) \vee (q_1 \,\&\, q_2 \,\&\, \cdots \,\&\, q_n) \vee \cdots \vee (r_1 \,\&\, r_2 \,\&\, \cdots \,\&\, r_n)$
- And, we notice that $(\forall x)(\exists y)Rxy$ has the following n-form:

  $\mathcal{Y}_n = (p_1 \vee q_1 \vee \cdots \vee r_1) \,\&\, (p_2 \vee q_2 \vee \cdots \vee r_2) \,\&\, \cdots \,\&\, (p_n \vee q_n \vee \cdots \vee r_n)$
- **Fact**. $\mathcal{X}_n \vDash \mathcal{Y}_n$, *for any* n. Each disjunct of $\mathcal{X}_n$ entails every conjunct of $\mathcal{Y}_n$. **Caution!** This ***doesn't*** show that $(\exists y)(\forall x)Rxy \vDash (\forall x)(\exists y)Rxy$!
- **Fact**. $\mathcal{Y}_n \nvDash \mathcal{X}_n$, for all $n > 1$. This can be shown (next slide) using only LSL reasoning. This ***does*** show that $(\forall x)(\exists y)Rxy \nvDash (\exists y)(\forall x)Rxy$.
- The moral is that our "informal" semantical approach to the quantifiers works for LMPL, since no infinite domains are required for $\nvDash$ in LMPL.

- However, our "informal" semantical approach breaks down for L2PL, since we sometimes need an infinite domain to establish $\nvDash$ in L2PL.
- In L2PL, if the "informal" method above reveals $p_n \nvDash q_n$ for *some* finite n, then it *does* follow that $p \nvDash q$. For instance, $\mathcal{Y}_2 \nvDash \mathcal{X}_2$ on the last slide:
  - $(Raa \vee Rab) \,\&\, (Rba \vee Rbb) \nvDash (Raa \,\&\, Rba) \vee (Rab \,\&\, Rbb)$
  - This is just an LSL problem with 4-atoms [$A = Raa$, $B = Rab$, $C = Rba$, $D = Rbb$]. Truth-tables will generate a counterexample.
- On the other hand, if (in L2PL) our "informal" method indicates (as above) that $p_n \vDash q_n$ for *all* finite n, this does *not* guarantee $p \vDash q$. *E.g.*:
  - $p = (\forall x)(\exists y)Rxy \,\&\, (\forall x)(\forall y)(\forall z)[(Rxy \,\&\, Ryz) \to Rxz]$.
  - $q = (\exists x)Rxx$.
- We showed above (informally) that $p_n \vDash q_n$ for *all* finite n. But, we also saw that there are infinite interpretations on which p is $\top$ but q is $\perp$.

Natural Deduction in L2PL

- The ND system we already have is sound and complete for L2PL (140A!).
- So, we're already in a position to do natural deductions in L2PL.
- Here are four examples, with solutions on subsequent slides:
  1. $(\exists y)(\forall x)Rxy \vdash (\forall x)(\exists y)Rxy$
     - This is the alternating quantifier example we just examined.
  2. $(\forall x)(\forall y)(\forall z)[(Rxz \& Ryz) \rightarrow Rxy], (\forall x)Rxx \vdash (\forall x)(\forall y)(Rxy \rightarrow Ryx)$
     - This is: Euclidean-ness + reflexivity entails symmetry.
  3. $(\forall x)(\forall y)(\forall z)[(Rxz \& Ryz) \rightarrow Rxy], (\forall x)Rxx$ $\vdash (\forall x)(\forall y)(Rxy \rightarrow Ryx)$
     - This is: Euclidean-ness + reflexivity entails transitivity.
  4. $(\forall x)Fx \leftrightarrow \sim(\exists x)(\exists y)Rxy \vdash (\exists x)(\forall y)(\forall z)(Fx \rightarrow \sim Ryz)$
  5. $(\forall x)(\forall y)(\forall z)(Rxy \rightarrow \sim Ryz) \vdash (\exists y)(\forall x)\sim Rxy$

L2PL Natural Deduction Problem #1

Problem is : (∃y)(∀x)Rxy ⊦ (∀x)(∃y)Rxy

1	(1)	(∃y)(∀x)Rxy	Premise
2	(2)	(∀x)Rxb	Ass (∃E)
2	(3)	Rab	2 ∀E
2	(4)	(∃y)Ray	3 ∃I
1	(5)	(∃y)Ray	1,2,4 ∃E
1	(6)	(∀x)(∃y)Rxy	5 ∀I

L2PL Natural Deduction Problem #2

Problem is : (∀x)(∀y)(∀z)((Rxz&Ryz)→Rxy), (∀x)Rxx ⊦ (∀x)(∀y)(Rxy→Ryx)

1	(1)	(∀x)(∀y)(∀z)((Rxz&Ryz)→Rxy)	Premise
2	(2)	(∀x)Rxx	Premise
3	(3)	Rab	Ass (→I)
1	(4)	(∀y)(∀z)((Rbz&Ryz)→Rby)	1 ∀E
1	(5)	(∀z)((Rbz&Raz)→Rba)	4 ∀E
1	(6)	(Rbb&Rab)→Rba	5 ∀E
2	(7)	Rbb	2 ∀E
2,3	(8)	Rbb&Rab	7,3 &I
1,2,3	(9)	Rba	6,8 →E
1,2	(10)	Rab→Rba	3,9 →I
1,2	(11)	(∀y)(Ray→Rya)	10 ∀I
1,2	(12)	(∀x)(∀y)(Rxy→Ryx)	11 ∀I

L2PL Natural Deduction Problem #3

Problem is : (∀x)(∀y)(∀z)((Rxz&Ryz)→Rxy), (∀x)Rxx ⊦ (∀x)(∀y)(∀z)((Rxy&Ryz)→Rxz)

1	(1)	(∀x)(∀y)(∀z)((Rxz&Ryz)→Rxy)	Premise
2	(2)	(∀x)Rxx	Premise
3	(3)	Rab&Rbc	Ass (→I)
1	(4)	(∀y)(∀z)((Raz&Ryz)→Ray)	1 ∀E
1	(5)	(∀z)((Raz&Rcz)→Rac)	4 ∀E
1	(6)	(Rab&Rcb)→Rac	5 ∀E
3	(7)	Rab	3 &E
1,2	(8)	(∀x)(∀y)(Rxy→Ryx)	1,2 [Problem #2!]
1,2	(9)	(∀y)(Rby→Ryb)	8 ∀E
1,2	(10)	Rbc→Rcb	9 ∀E
3	(11)	Rbc	3 &E
1,2,3	(12)	Rcb	10,11 →E
1,2,3	(13)	Rab&Rcb	7,12 &I
1,2,3	(14)	Rac	6,13 →E
1,2	(15)	(Rab&Rbc)→Rac	3,14 →I
1,2	(16)	(∀z)((Rab&Rbz)→Raz)	15 ∀I
1,2	(17)	(∀y)(∀z)((Ray&Ryz)→Raz)	16 ∀I
1,2	(18)	(∀x)(∀y)(∀z)((Rxy&Ryz)→Rxz)	17 ∀I

L2PL Natural Deduction Problem #4

Problem is : (∀x)Fx, ~(∃x)(∃y)Rxy ⊢ (∃x)(∀y)(∀z)(Fx→~Ryz)

1	(1)	(∀x)Fx	Premise
2	(2)	~(∃x)(∃y)Rxy	Premise
3	(3)	Fa	Ass (→I)
2	(4)	(∀x)~(∃y)Rxy	2 SI (QS)
2	(5)	~(∃y)Rby	4 ∀E
2	(6)	(∀y)~Rby	5 SI (QS)
2	(7)	~Rbc	6 ∀E
2	(8)	Fa→~Rbc	3,7 →I
2	(9)	(∀z)(Fa→~Rbz)	8 ∀I
2	(10)	(∀y)(∀z)(Fa→~Ryz)	9 ∀I
2	(11)	(∃x)(∀y)(∀z)(Fx→~Ryz)	10 ∃I

L2PL Natural Deduction Problem #5

Problem is : (∀x)(∀y)(∀z)(Rxy→~Ryz) ⊢ (∃y)(∀x)~Rxy

1	(1)	(∀x)(∀y)(∀z)(Rxy→~Ryz)	Premise
2	(2)	~(∃y)(∀x)~Rxy	Ass (~I)
2	(3)	(∀y)~(∀x)~Rxy	2 SI (QS)
2	(4)	~(∀x)~Rxa	3 ∀E
2	(5)	(∃x)~~Rxa	4 SI (QS)
6	(6)	~~Rba	Ass (∃E)
2	(7)	~(∀x)~Rxb	3 ∀E
2	(8)	(∃x)~~Rxb	7 Taut.
9	(9)	~~Rcb	Ass (∃E)
1	(10)	(∀y)(∀z)(Rcy→~Ryz)	1 ∀E
1	(11)	(∀z)(Rcb→~Rbz)	10 ∀E
1	(12)	Rcb→~Rba	11 ∀E
9	(13)	Rcb	9 DN
1,9	(14)	~Rba	12,13 →E
6	(15)	Rba	6 DN
1,6,9	(16)	⋏	14,15 ~E
1,2,6	(17)	⋏	8,9,16 ∃E
1,2	(18)	⋏	5,6,17 ∃E
1	(19)	~~(∃y)(∀x)~Rxy	2,18 ~I
1	(20)	(∃y)(∀x)~Rxy	19 DN

Overview of the Course I

- Deductive Logic provides *formal theories* of *validity*. The deductive logician aims to *theoretically ground* our *informal* validity notion.
- In English, there are various argument forms or patterns that are intuitively or informally valid. We began with *sentential* forms like:

Dr. Ruth is a man.
(1) If Dr. Ruth is a man, then Dr. Ruth is 10 feet tall.
∴ Dr. Ruth is 10 feet tall.

- Intuitively, (1)'s conclusion *follows-from* its premises. *If* the premises of (1) *were* all true, *then* (1)'s conclusion would also *have to be* true.
- Our first logical theory (LSL) correctly classifies this argument form (and many other valid English forms) as *valid.* A "success story" for LSL:

p.
(1_{LSL}) If p, then q.
∴ q.

Overview of the Course II

- However, there are many English arguments that are (intuitively, or "absolutely") valid, but their LSL forms are *not* valid. For instance:

(2) Socrates is wise.
∴ Someone is wise.

- Intuitively, argument (2) is ("absolutely") *valid*. But, if we try to translate this argument into LSL, we get the following *in*valid LSL form:

(2_{LSL}) p.
∴ q.

- This is why we moved to the richer language LMPL, which subsumes LSL, and which adds additional structure that allows us to capture (2):

(2_{LMPL}) Ws
∴ $(\exists x)Wx$

- Moreover, as we have seen, there are other valid English arguments that are beyond the reach of even the richer formal logical theory LMPL. *E.g.*:

(3) Brutus killed Caesar.
∴ Brutus killed someone and someone killed Caesar.

- If we were to symbolize this argument using LMPL, we would end-up with something like the following *in*valid LMPL argument form:

(3_{LMPL}) Kb
$\therefore (\exists x)Bx \,\&\, (\exists y)Ky$

Where Kx: x killed Caesar, Bx: Brutus killed x, and b: Brutus.

- The still richer language L2PL introduces 2-place predicates, such as Kxy: x killed y. With this relation in hand, we can capture (3) as:

(3_{L2PL}) Kbc
$\therefore (\exists x)Kbx \,\&\, (\exists y)Kyc$

- Of course, there are arguments beyond even L2PL's reach…

Beyond L2PL I — Full First-Order Logic (LFOL)

- The full theory of first-order logic (LFOL) includes L2PL, plus n-place predicates, the identity relation =, and also function symbols.
- LFOL can capture even more valid arguments than L2PL. For instance, LFOL can capture arguments like the following mathematical one:

(4) $2 + 4 = 6$
$3 \times 2 = 6$
$\therefore 2 + 4 = 3 \times 2$

- Indeed, LFOL can capture just about any argument in just about any branch of modern mathematics. That's a lot of expressive power.
- In PHIL 140A, we study the full theory of first-order logic (LFOL). There, we give a semantics for LFOL, and we show that there is a sound and complete proof theory for LFOL (but, no decision procedure for $\vDash$!).
- Of course, even full first-order logic (LFOL) has its limitations…

Beyond L2PL II — Second-Order Deductive Logic

- Some arguments involve quantification over not only objects but *properties*. These arguments are *second-order* and ∴ beyond LFOL.
- Leibniz (sometimes) talked as if the following argument were valid:

(5) a and b have exactly the same (monadic) properties.
∴ a and b are identical.

- In second-order logic (SOL), (5) would be formalized as follows:

(5_{SOL}) $(\forall P)(Pa \leftrightarrow Pb)$.
$\therefore a = b$.

- Note that the premise of (5) quantifies over (monadic) *predicates*.
- This is something that LFOL is not designed to do.
- We could also have an SOL which allows quantification over *relations*.
- Second-order logic is beyond 140A. It is touched upon (a little) in 140B.

Beyond L2PL III — Non-Classical Deductive Logics

- All the logics I've mentioned are *classical* deductive logics. Not all logicians think classical logics capture our intuitive validity notions.
- Classical logics all share the following two properties:
  - (i) All arguments with contradictory premises (*e.g.*, $p \,\&\, {\sim}p$) are valid.
  - (ii) All arguments with tautological conclusions (*e.g.*, $p \vee {\sim}p$) are valid.
- Some logicians think (i) and/or (ii) are *counterexamples* to the classical theory of validity (as an explication of our informal "following-from").
- Such logicians propose alternative formal theories of validity ($\vDash^\star$).
- Usually, non-classical logicians reject the classical (truth-functional) theory of the *conditional*. They adopt a non-classical conditional ($\rightarrow^\star$) which obeys constraints like the deduction theorem (relative to $\vDash^\star$).

$$p \vDash^\star q \text{ if and only if } \vDash p \rightarrow^\star q$$

- These and other fundamental philosophical questions about the nature of logic are addressed in our Philosophical Logic course (PHIL 142).

Beyond L2PL IV — Inductive Logics

- Intuitively, not all "logically good" arguments are deductively valid. Some invalid arguments seem (intuitively) logically *better than* others:

(6) p. Someone is wise.
$\therefore q$. Socrates is wise.

(7) r. Someone is either wise or unwise.
$\therefore q$. Socrates is wise.

- *Inductive* logic should *theoretically ground* our intuition that (6) is a *logically stronger* argument than (7) is. Neither argument is *valid*.
- More ambitiously, an inductive logician might aim for a theory of "the *degree* to which the premises of an argument *confirm* its conclusion".
- This ambitious project would aim to characterize a *function* $\mathfrak{c}(\mathscr{C},\mathscr{P})$. And, an intuitive requirement would be that this function be such that:

$$\mathfrak{c}(q,p) > \mathfrak{c}(q,r)$$

- PHIL 148 is about probability and *inductive logic*. There, we explain how *probabilities* can be used to define various $\mathfrak{c}$-functions.

Brief Review for Final

- The exam will contain 8 questions (same as the sample).
  - No L2PL interpretation *construction* problems on main exam.
  - There will also be an L2PL extra-credit problem. This problem will involve symbolization of an L2PL argument, and either interpretation construction or natural deduction.
  - Only try the extra-credit if you've got everything else nailed down.
- You will be given all natural deduction rules, but no truth-table definitions. The final rules handout is already posted online.
- You will be given 3 hours to complete the exam (you probably won't need it, but it's not a bad idea to go over things twice).
- Make sure to bring (several) blue books and a writing implement (preferably, a pencil). Turn in all materials at end.

Farewell

Thanks for a great 6 weeks!

Good luck on your finals!

Have a great summer!